On the sound field of a resilient disk in free space

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Radiation characteristics are calculated for a circular planar sound source in free space with a uniform surface pressure distribution, which can be regarded as a freely suspended membrane with zero mass and stiffness. This idealized dipole source is shown to have closed form solutions for its far-field pressure response and radiation admittance. The latter is found to have a simple mathematical relationship with the radiation impedance of a rigid piston in an infinite baffle. Also, a single expansion is derived for the near-field pressure field, which degenerates to a closed form solution on the axis of symmetry. From the normal gradient of the surface pressure, the surface velocity is calculated. The near-field expression is then generalized to an arbitrary surface pressure distribution. It is shown how this can be used as a simplified solution for a rigid disk in free space or a more realistic sound source such as pre-tensioned membrane in free space with non-zero mass and a clamped rim. [DOI: 10.1121/1.2839891]

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I. INTRODUCTION

The resilient disk in free space is the dipole complement of the rigid disk in an infinite baffle. Together with a few variants,[1] these are the only axisymmetric planar sources that yield compact closed-form solutions for their axial and far-field pressure responses and radiation impedances. Interchanging the boundary conditions leads to another complementary pair of axisymmetric planar sources, namely the resilient disk in an infinite baffle and rigid disk in free space. These are slightly more complicated, but the solutions are also applicable to diffraction problems using Babinet's principle,[2] as modified by Bouwkamp.[3] The reason for the extra complexity is the mixture of velocity and pressure boundary conditions in the plane of the disk. In a baffle, the resilient disk has a uniform driving pressure across its surface and zero velocity beyond its rim. Early solutions to this problem involved iterative methods based upon oblate spheroidal wave functions.[3,4] An alternative approach[5] is to use the King integral, which is similar to the Rayleigh[6] integral except that the Green's function in cylindrical coordinates (which has been termed the Lamb–Sommerfeld integral) is used, as opposed to the rotationally symmetric spherical Green's function. The disk velocity distribution can be represented by a trial function which itself is based upon the solution to the free space wave equation in oblate spheroidal coordinates.

The rigid disk in free space, conversely, has uniform velocity across its surface and zero pressure beyond its rim, so that a similar approach[7] can be applied, but using a trial function for the disk pressure distribution instead. Sets of simultaneous equations are then developed and solved numerically for the unknown trial function coefficients. A similar approach can also be applied to fluid-structure coupled problems,[8–12] where neither the disk velocity nor pressure distributions are uniform, and so the coupled disk and free space wave equations have to be solved simultaneously.

The simplest monopole planar source is the rigid disk (or piston) in an infinite baffle, which has a velocity boundary condition on both its face and the surrounding baffle. Remarkably, this was first derived by Rayleigh[6] before the direct radiator loudspeaker had even been invented,[13] yet it has been widely accepted as a model for such when mounted in a box near a wall or, even better, mounted directly in a wall as commonly found in recording studios. The model is useful in the frequency range up to the first diaphragm breakup mode.

Unfortunately, the Rayleigh integral is not particularly amenable to numerical calculation of the near-field pressure, especially at high frequencies. The integrand is oscillatory and the Green's function is singular at the source. Hence there has been a strong motivation to find alternative methods, especially those using fast converging expansions. A useful review of previous literature relating to the baffled planar piston was provided by Harris,[14] which includes some early movable origin schemes, whereby the origin of the coordinate system was moved to the same radial distance as the observation point when projected onto the plane of the radiator. Later, Hasagawa *et al.*[15] moved the origin axially in front of the radiator in order to achieve convergence in the immediate near field. Recently, Mast and Yu[16] have supplied an elegant single-expansion solution in a similar manner, but locking the origin in the same plane as the observation point. It is shown in this paper how a similar expansion can be obtained for the resilient disk. Also, apodized radiators have been studied by Kelly and McGough.[17]

The resilient disk in free space is the simplest dipole planar source, having a uniform driving pressure across its face and zero pressure extending beyond its rim. It can be used as an approximate model for unbaffled loudspeakers of the electrostatic or planar magnetic type, in which it is assumed that a perfectly uniform driving pressure is applied to a very light flexible membrane diaphragm in free space.

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Walker[18] pointed out that such a source is acoustically transparent, in that it does not disturb the field around it, and used this idealized model to derive the far-field on-axis pressure response of an electrostatic loudspeaker, which provides a useful approximation over the loudspeaker's working range. However, it should be noted that the model assumes a freely suspended membrane, whereas in reality it is usually clamped at the rim, which effectively removes the singularity from the rim of the idealized model.

Bouwkamp[3] solved the real radiation admittance (or conductance, aka transmission coefficient), but the imaginary radiation admittance (or susceptance) has remained hitherto unsolved. An alternative derivation to that of Bouwkamp for the conductance is provided here, which is based upon the dipole version of the King[19] integral. Although this approach has previously been used by Morse and Ingard,[20] they did not solve the equations for the conductance or susceptance, but presented approximate solutions based upon an oscillating rigid sphere, together with the correct far-field expression. Here, a formal derivation is presented, using known identities, which shows a simple relationship between the admittance of a resilient disk and the impedance of a rigid disk.

In Sec. II of this paper, the boundary conditions of the problem are set out, after which a solution to the free space wave equation using the dipole King integral is presented in Sec. III, following the approach of Morse *et al.* In Sec. IV the radiation conductance and susceptance are rigorously derived and it is shown how these relate to the resistance and reactance of a rigid disk in an infinite baffle. Some remarks on an earlier attempt by the author to solve the susceptance integral by symbolic computation are also included.

In Sec. V, a solution to the free space equation using the dipole Rayleigh integral is derived, where the Green's function is expanded using the Gegenbauer addition theorem (or multipole expansion). This leads to a single-expansion expression for the pressure field when the distance from the center of the disk to the observation point is greater than the disk's radius. A solution for a planar axisymmetric source with an arbitrary surface pressure distribution is also included. In Sec. VI, the paraxial pressure field is derived, which converges in the immediate near field and is again a single expansion, reducing to a single term, or closed-form solution, on the axis of symmetry. From the paraxial solution, the expression for the surface-velocity distribution, given in Sec. VII, is fairly straightforward to derive due to the fact that the paraxial solution is in cylindrical coordinates. This makes it fairly easy to take the normal derivative of the pressure with respect to the axial ordinate at the surface of the disk. Finally, in Sec. VII, the expression for the far-field pressure is presented. Although this expression is nothing new, it is interesting to compare the beam pattern with that of a rigid disk and it is shown that, in the case of an electrostatic loudspeaker, this gives the same on-axis pressure as Walker's equation.

The general aim of this paper is to provide a full set of radiation characteristics of the resilient disk in free space and to show that they generally have simple relationships with those of a rigid disk in an infinite baffle.

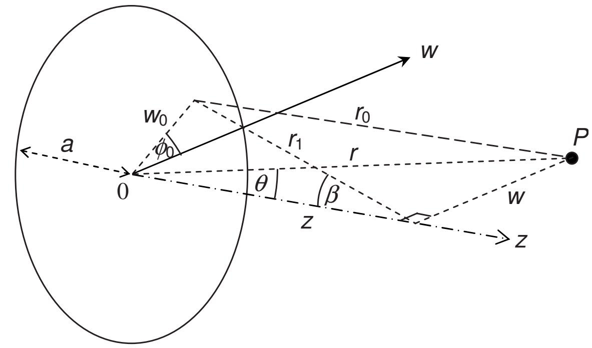

FIG. 1. Geometry of the disk.

II. BOUNDARY CONDITIONS

The infinitesimally thin resilient disk shown in Fig. 1 has a radius a and lies in the w plane with its center at the origin. Due to axial symmetry, the tangential ordinate ϕ of the coordinate system for the observation point P can be ignored. Hence it is simply defined, in spherical coordinates, by the radial and azimuthal ordinates r and θ, respectively or, in cylindrical coordinates, by the radial and axial ordinates w and z, respectively. The infinitesimally thin membrane-like resilient disk is assumed to be perfectly flexible, has zero mass, and is free at its perimeter. It is driven by a uniformly distributed harmonically varying pressure $\tilde{p}_0$ and thus radiates sound from both sides into a homogeneous loss-free acoustic medium. In fact, there need not be a disk present at all and instead the driving pressure could be acting upon the air particles directly. However, for expedience, the area over which this driving pressure is applied shall be referred to as a disk from here onwards. The pressure field on one side of the xy plane is the symmetrical "negative" of that on the other, so that

$$\tilde{p}(w,z) = -\tilde{p}(w,-z). \tag{1}$$

Consequently, there is a Dirichlet boundary condition in the plane of the disk where these equal and opposite fields meet

$$\tilde{p}(w,0) = 0, \quad a < w \leqslant \infty . \tag{2}$$

On the front and rear surfaces of the disk, the pressures are $\tilde{p}_+$ and $\tilde{p}_-$, respectively, which are given by

$$\tilde{p}_+(w_0) = -\tilde{p}_-(w_0) = \tilde{p}_0/2, \quad 0 \leqslant w_0 \leqslant a \tag{3}$$

and k is the wave number given by $k=\omega/c=2\pi/\lambda$, where ω is the angular frequency of excitation, ρ is the density of the surrounding medium, c is the speed of sound in that medium, and λ is the wavelength. The annotation $\tilde{}$ denotes a harmonically time-varying quantity.

III. SOLUTION OF THE FREE-SPACE WAVE EQUATION

Using the dipole King integral,[19] the pressure distribution is defined by

$$\tilde{p}(w,z) = \int_0^{2\pi}\int_0^a (\tilde{p}_+(w_0) - \tilde{p}_-(w_0)) \times \frac{\partial}{\partial z_0} g(w,z|w_0,z_0)\Big|_{z_0=0+} w_0 dw_0 d\phi_0, \quad (4)$$

where the Green's function[20] is defined, in cylindrical coordinates, by

$$g(w,z|w_0,z_0) = \frac{i}{4\pi}\int_0^\infty J_0(\mu w)J_0(\mu w_0)\frac{\mu}{\sigma}e^{-i\sigma|z-z_0|}d\mu, \quad (5)$$

where

$$\sigma = \begin{cases} \sqrt{k^2-\mu^2}, & 0 \leqslant \mu < k \\ -i\sqrt{\mu^2-k^2}, & \mu > k \end{cases} \quad (6)$$

and J_n is the Bessel function of the first kind. Substituting Eqs. (3), (5), and (6) in Eq. (4) and integrating over the surface of the disk yields

$$\tilde{p}(w,z) = \frac{a\tilde{p}_0}{2}\int_0^\infty J_1(\mu a)J_0(\mu w)e^{-i\sigma z}d\mu, \quad (7)$$

where the following identity[22] has been used

$$\int_0^a J_0(\mu w_0)w_0 dw_0 = \frac{a}{\mu}J_1(a\mu). \quad (8)$$

IV. RADIATION ADMITTANCE

A. Admittance as an integral expression

The disk velocity $\tilde{u}_0(w)$ can be derived using the following relationship for the normal pressure gradient:

$$\tilde{u}_0(w) = \frac{i}{k\rho c}\frac{\partial}{\partial z}\tilde{p}(w,z)\Big|_{z=0+} = \frac{a\tilde{p}_0}{2k\rho c}\int_0^\infty J_1(\mu a)J_0(\mu w)\sigma d\mu. \quad (9)$$

For small k, this reduces to the Weber–Schafheitlin integral[21,22]

$$\begin{aligned}\tilde{u}_0(w)|_{k\to 0} &= \frac{ia\tilde{p}_0}{2k\rho c}\int_0^\infty J_1(\mu a)J_0(\mu w)\mu d\mu \\ &= \frac{i\tilde{p}_0}{2ka\rho c}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};1;\frac{w^2}{a^2}\right) \\ &= \frac{i\tilde{p}_0 \boldsymbol{E}(w^2/a^2)}{\pi ka\rho c}\left(1-\frac{w^2}{a^2}\right)^{-1} \\ &\approx \frac{i\tilde{p}_0}{2ka\rho c}\left\{1-\left(1-\frac{2}{\pi}\right)\frac{w^3}{a^3}\right\}\left(1-\frac{w^2}{a^2}\right)^{-1},\end{aligned} \quad (10)$$

where $\boldsymbol{E}$ is the complete elliptic integral of the second kind. Hence there is a singularity at the rim. Integrating the velocity from Eq. (9) over the area of the disk provides the total volume velocity $\tilde{U}_0$ as follows:

$$\tilde{U}_0 = \int_0^{2\pi}\int_0^a \tilde{u}_0(w)w dw d\phi = \frac{\pi a^2\tilde{p}_0}{k\rho c}\int_0^\infty J_1^2(\mu a)\frac{\sigma}{\mu}d\mu, \quad (11)$$

where Eq. (8) has again been used. The acoustic radiation admittance is then given by

$$y_{ar} = \frac{\tilde{U}_0}{\tilde{p}_0} = \frac{S\tilde{u}_0}{\tilde{p}_0} = \frac{S}{2\rho c}(G_R(k) - iB_R(k)), \quad (12)$$

where G_R is the normalized *conductance* given by

$$G_R(k) = \frac{2}{k}\int_0^k \frac{\sqrt{k^2-\mu^2}}{\mu}J_1^2(\mu a)d\mu, \quad (13)$$

B_R is the normalized *susceptance* given by

$$B_R(k) = \frac{2}{k}\int_k^\infty \frac{\sqrt{\mu^2-k^2}}{\mu}J_1^2(\mu a)d\mu, \quad (14)$$

and $S=\pi a^2$ is the surface area of the disk.

B. Solution of the real integral

Substituting $\mu=k\sin\theta$ and $\zeta=ka$ in Eq. (13) yields

$$G_R(\zeta) = 2\int_0^{\pi/2}\frac{\cos^2\theta}{\sin\theta}J_1^2(\zeta\sin\theta)d\theta, \quad (15)$$

which, after differentiating with respect to ζ, gives

$$\frac{d}{d\zeta}G_R(\zeta) = 2\int_0^{\pi/2} J_1(\zeta\sin\theta) \times (J_0(\zeta\sin\theta) - J_2(\zeta\sin\theta))\cos^2\theta d\theta. \quad (16)$$

Using the following identities[23]

$$J_1(\zeta\sin\theta)J_0(\zeta\sin\theta) = \frac{2}{\pi}\int_0^{\pi/2}\cos\phi J_1(2\zeta\sin\theta\cos\phi)d\phi \quad (17)$$

and

$$J_1(\zeta\sin\theta)J_2(\zeta\sin\theta) = -\frac{2}{\pi}\int_0^{\pi/2}\cos 3\phi J_1(2\zeta\sin\theta\cos\phi)d\phi \quad (18)$$

together with[22]

$$\cos\phi + \cos 3\phi = 2\cos\phi\cos 2\phi, \quad (19)$$

Eq. (16) becomes

$$\frac{d}{d\zeta}G_R(\zeta) = \frac{8}{\pi}\int_0^{\pi/2}\cos\phi\cos 2\phi \times \int_0^{\pi/2}J_1(2\zeta\cos\phi\sin\theta)\cos^2\theta d\theta d\phi. \quad (20)$$

The integral over θ is split into two using $\cos^2\theta=1-\sin^2\theta$ and then solved with the help of the following identities:[22]

$$\int_0^{\pi/2} J_1(\psi \sin\theta) d\theta = \sqrt{\frac{\pi}{2\psi}} \mathbf{H}_{1/2}(\psi) = \frac{1-\cos\psi}{\psi}, \quad (21)$$

$$\int_0^{\pi/2} J_1(\psi \sin\theta) \sin^2\theta d\theta = \sqrt{\frac{\pi}{2\psi}} J_{3/2}(\psi) = \frac{\sin\psi}{\psi^2} - \frac{\cos\psi}{\psi}, \quad (22)$$

where $\mathbf{H}_n$ is the Struve function and $\psi=2\zeta\cos\phi$, so that

$$\frac{d}{d\zeta} G_R(\zeta) = \frac{8}{\pi} \int_0^{\pi/2} \cos\phi \cos 2\phi \left(\frac{1}{\psi} - \frac{\sin\psi}{\psi^2} \right) d\phi. \quad (23)$$

The integral of the first term in the bracket vanishes and using the identity $\cos 2\phi = 2\cos^2\phi - 1$, gives

$$\frac{d}{d\zeta} G_R(\zeta) = \frac{2}{\pi\zeta^2} \int_0^{\pi/2} \frac{\sin(2\zeta\cos\phi)}{\cos\phi} d\phi - \frac{4}{\pi\zeta^2} \int_0^{\pi/2} \sin(2\zeta\cos\phi) \cos\phi d\phi. \quad (24)$$

The first integral in Eq. (24) is differentiated to give[22]

$$\frac{d}{d\zeta} \int_0^{\pi/2} \frac{\sin(2\zeta\cos\phi)}{\cos\phi} d\phi = 2\int_0^{\pi/2} \cos(2\zeta\cos\phi) d\phi = \pi J_0(2\zeta). \quad (25)$$

The second is solved using the identity[22]

$$\int_0^{\pi/2} \sin(2\zeta\cos\phi) \cos\phi d\phi = \frac{\pi}{2} J_1(2\zeta). \quad (26)$$

Hence

$$\frac{d}{d\zeta} G_R(\zeta) = \frac{2}{\zeta^2} \left(\int J_0(2\zeta) d\zeta - J_1(2\zeta) \right), \quad (27)$$

or using the product rule

$$\zeta \frac{d}{d\zeta} G_R(\zeta) = \frac{2}{\zeta} \left(\int J_0(2\zeta) d\zeta - J_1(2\zeta) \right) = \frac{d}{d\zeta} \zeta G_R(\zeta) - G_R(\zeta). \quad (28)$$

Let the solution be

$$G_R(\zeta) = 1 - \frac{J_1(2\zeta)}{\zeta} - \frac{2}{\zeta} \left(\int J_0(2\zeta) d\zeta - J_1(2\zeta) \right) = 1 + \frac{J_1(2\zeta)}{\zeta} - \frac{2}{\zeta} \int J_0(2\zeta) d\zeta. \quad (29)$$

Then[22]

$$\frac{d}{d\zeta} \zeta G_R(\zeta) = 1 + J_0(2\zeta) - J_2(2\zeta) - 2J_0(2\zeta) = 1 - \frac{J_1(2\zeta)}{\zeta}, \quad (30)$$

which is the radiation resistance of a rigid disk in an infinite baffle.[7,19] It can easily be seen that Eqs. (29) and (30) satisfy Eq. (28). With help from the following identity:[24]

$$\int J_0(2\zeta) d\zeta = \int_0^{\zeta} J_0(2\zeta) d\zeta = \zeta J_0(2\zeta) + \pi\zeta \times \frac{J_1(2\zeta)\mathbf{H}_0(2\zeta) - J_0(2\zeta)\mathbf{H}_1(2\zeta)}{2}, \quad (31)$$

the final solution is then given by

$$G_R(ka) = 1 + \frac{J_1(2ka)}{ka} - 2J_0(2ka) - \pi(J_1(2ka)\mathbf{H}_0(2ka) - J_0(2ka)\mathbf{H}_1(2ka)) \approx \frac{k^2a^2}{6}, \quad ka < 0.5. \quad (32)$$

C. Solution of the imaginary integral

Substituting $\mu = k\sin\theta$ and $\zeta = ka$ in Eq. (14) yields

$$B_R(\zeta) = 2i \int_{(\pi/2)+i0}^{(\pi/2)+i\infty} \frac{\cos^2\theta}{\sin\theta} J_1^2(\zeta\sin\theta) d\theta, \quad (33)$$

which, after differentiating with respect to ζ, gives

$$\frac{d}{d\zeta} B_R(\zeta) = 2i \int_{(\pi/2)+i0}^{(\pi/2)+i\infty} J_1(\zeta\sin\theta) \times (J_0(\zeta\sin\theta) - J_2(\zeta\sin\theta)) \cos^2\theta d\theta. \quad (34)$$

Using the identities of Eqs. (17)–(19), Eq. (34) becomes

$$\frac{d}{d\zeta} B_R(\zeta) = \frac{8i}{\pi} \int_0^{\pi/2} \cos\phi \cos 2\phi \times \int_{(\pi/2)+i0}^{(\pi/2)+i\infty} J_1(2\zeta\cos\phi\sin\theta) \cos^2\theta d\theta d\phi \quad (35)$$

and let $t = \sin\theta$ so that

$$\frac{d}{d\zeta} B_R(\zeta) = -\frac{8}{\pi} \int_0^{\pi/2} \cos\phi \cos 2\phi \times \int_1^{\infty} J_1(2\zeta\cos\phi t) \sqrt{t^2-1} dt d\phi. \quad (36)$$

The integral over t is then solved with the help of the following identity:[22]

$$\int_1^{\infty} J_1(\psi t) \sqrt{t^2-1} dt = \frac{\cos\psi}{\psi^2}, \quad (37)$$

where $\psi = 2\zeta\cos\phi$, so that

$$\frac{d}{d\zeta} B_R(\zeta) = -\frac{2}{\pi\zeta^2} \int_0^{\pi/2} \frac{\cos 2\phi \cos\psi}{\cos\phi} d\phi. \quad (38)$$

Using the identity $\cos 2\phi = 2\cos^2\phi - 1$, gives

$$\frac{d}{d\zeta}B_R(\zeta) = \frac{2}{\pi\zeta^2}\int_0^{\pi/2}\frac{\cos(\psi)}{\cos\,\phi}d\phi - \frac{4}{\pi\zeta^2}\int_0^{\pi/2}\cos(\psi)\cos\,\phi d\phi. \tag{39}$$

The first integral in Eq. (39) is differentiated to give[22]

$$\frac{d}{d\zeta}\int_0^{\pi/2}\frac{\cos(2\zeta\cos\,\phi)}{\cos\,\phi}d\phi = -2\int_0^{\pi/2}\sin(2\zeta\cos\,\phi)d\phi = -\pi\mathbf{H}_0(2z). \tag{40}$$

The second is solved using the identity[22]

$$\int_0^{\pi/2}\cos(\psi)\cos\,\phi d\phi = \frac{d}{d\zeta}\int_0^{\pi/2}\int\cos(2\zeta\cos\,\phi)\cos\,\phi d\zeta d\phi = \frac{d}{d\zeta}\int_0^{\pi/2}\frac{\sin(2\zeta\cos\,\phi)}{2}d\phi = \frac{d}{d\zeta}\frac{\pi\mathbf{H}_0(2\zeta)}{4} = \frac{\pi\mathbf{H}_{-1}(2\zeta)}{2}. \tag{41}$$

Hence

$$\frac{d}{d\zeta}B_R(\zeta) = -\frac{2}{\zeta^2}\left(\int\mathbf{H}_0(2\zeta)d\zeta + \mathbf{H}_{-1}(2\zeta)\right), \tag{42}$$

or using the product rule

$$\zeta\frac{d}{d\zeta}B_R(\zeta) = -\frac{2}{\zeta}\left(\int\mathbf{H}_0(2\zeta)d\zeta + \mathbf{H}_{-1}(2\zeta)\right) = \frac{d}{d\zeta}\zeta B_R(\zeta) - B_R(\zeta). \tag{43}$$

Let the solution be

$$B_R(\zeta) = \frac{\mathbf{H}_1(2\zeta)}{\zeta} + \frac{2}{\zeta}\left(\int\mathbf{H}_0(2\zeta)d\zeta + \mathbf{H}_{-1}(2\zeta)\right) = \frac{4}{\pi\zeta} - \frac{\mathbf{H}_1(2\zeta)}{\zeta} + \frac{2}{\zeta}\int\mathbf{H}_0(2\zeta)d\zeta. \tag{44}$$

Then[22]

$$\frac{d}{d\zeta}\zeta B_R(\zeta) = \mathbf{H}_0(2\zeta) + \mathbf{H}_2(2\zeta) - \frac{4\zeta}{3\pi} = \frac{\mathbf{H}_1(2\zeta)}{\zeta}, \tag{45}$$

which is the radiation reactance of a rigid disk in an infinite baffle.[7,19] It can easily be seen that Eqs. (44) and (45) satisfy Eq. (43). With help from the following identity[22] (after substituting $\zeta = bx^{1/2}$, $\nu=0$, and $\mu=\lambda=0$):

$$\int_0^b\mathbf{H}_0\left(\frac{a\zeta}{b}\right)d\zeta = \frac{ab}{\pi}\,{}_2F_3\left(1,1;\frac{3}{2},\frac{3}{2},2;-\frac{a^2}{4}\right), \tag{46}$$

where ${}_pF_q$ is the hypergeometric function. Then, letting $a=2\zeta$ and $b=\zeta$, leads to

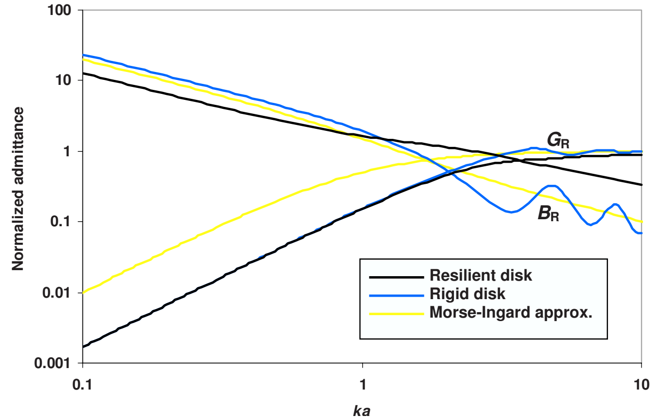

FIG. 2. (Color online) Normalized radiation admittances of the resilient and rigid disks in free space.

$$\int\mathbf{H}_0(2\zeta)d\zeta = \int_0^\zeta\mathbf{H}_0(2\zeta)d\zeta = \frac{2\zeta^2}{\pi}\,{}_2F_3\left(1,1;\frac{3}{2},\frac{3}{2},2;-\zeta^2\right), \tag{47}$$

so that the final solution is then given by

$$B_R(ka) = \frac{4}{\pi ka} - \frac{\mathbf{H}_1(2ka)}{ka} + \frac{4ka}{\pi}\,{}_2F_3\left(1,1;\frac{3}{2},\frac{3}{2},2;-k^2a^2\right) \approx \frac{4}{\pi ka}, \quad ka < 0.5. \tag{48}$$

The conductance G_R and reactance B_R are plotted in Fig. 2, along with the conductance and reactance of a rigid disk in free space for comparison. A third pair of curves shows the conductance and reactance of an oscillating sphere, used as approximations by Morse and Ingard,[20] whereby $G_R(ka) = k^2a^2/(1+k^2a^2)$ and $B_R(ka) = (2+k^2a^2)/(ka+k^3a^3)$. The same results are shown as impedances in Fig. 3.

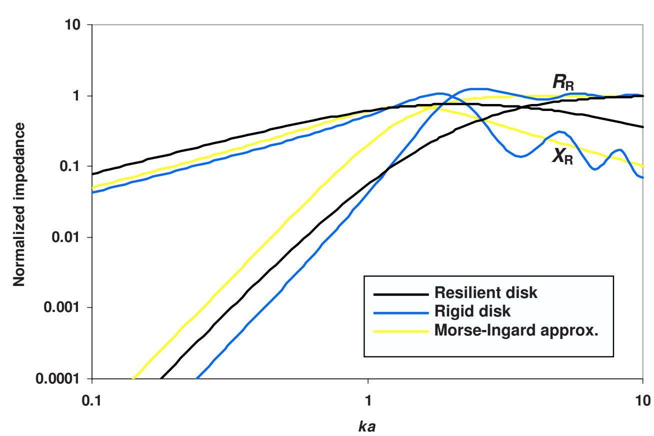

FIG. 3. (Color online) Normalized radiation impedances of the resilient and rigid disks in free space.

D. Relationship between a resilient disk in free space and a rigid disk in an infinite baffle

Suppose that the radiation resistance and reactance of a rigid disk in an infinite baffle are denoted by $R_R(ka)$ and X_R, respectively, then

$$\frac{d}{d(ka)}kaG_R(ka) = R_R(ka) = 1 - \frac{J_1(2ka)}{ka}, \tag{49}$$

or

$$G_R(ka) = \frac{1}{ka}\int R_R(ka)d(ka) \tag{50}$$

and

$$\frac{d}{d(ka)}kaB_R(ka) = X_R(ka) = \frac{\mathbf{H}_1(2ka)}{ka}, \tag{51}$$

or

$$B_R(ka) = \frac{1}{ka}\left(\int X_R(ka)d(ka) + \frac{4}{\pi}\right), \tag{52}$$

where G_R and B_R are the radiation conductance and susceptance, respectively, of a resilient disk in free space as defined in Eqs. (32) and (48). The constant of integration $4/\pi$ in Eq. (52) comes from Eq. (44). The low-frequency asymptotic values of disks in general are related by

$$R_R(\text{baffled resilient}) = R_R(\text{baffled rigid}) = k^2a^2/2, \tag{53}$$

$$X_R(\text{baffled resilient}) = X_R(\text{unbaffled resilient}) = \pi ka/4, \tag{54}$$

$$G_R(\text{unbaffled rigid}) = G_R(\text{unbaffled resilient}) = k^2a^2/6, \tag{55}$$

$$B_R(\text{unbaffled rigid}) = 2B_R(\text{baffled rigid}) = 3\pi/(4ka). \tag{56}$$

E. Some remarks on the solution by symbolic computation

This section reports an earlier attempt that had been made to solve the susceptance integral by symbolic computation,[21] but the result contained some erroneous terms and, without prior knowledge of the correct solution, it was impossible to tell which terms were correct and which were not. After substituting $\mu = kt$ in Eq. (14), the result of the symbolic computation[21] (after gathering a few terms) is

$$\begin{aligned} B_R(ka) &= 2\int_1^\infty \frac{\sqrt{t^2-1}}{t}J_1^2(kat)dt \\ &= 2\frac{2-\log(8ka)-\gamma}{\pi ka} \\ &\quad + \frac{4ka}{3\pi}{}_2F_3\left(1,1;\frac{3}{2},2,\frac{5}{2};-k^2a^2\right), \end{aligned} \tag{57}$$

which contains two erroneous negative terms. However, the correct form can be obtained from Eq. (52), which, after symbolic computation, gives

$$B_R(ka) = \frac{4}{\pi ka} + \frac{4ka}{3\pi}{}_2F_3\left(1,1;\frac{3}{2},2,\frac{5}{2};-k^2a^2\right). \tag{58}$$

Admittedly this is a particularly difficult integral to compute due to the oscillatory and slowly converging nature of the integrand. These days, symbolic computation is fairly reliable, but it is always worth checking the results numerically where possible, although in this case that is not so easy to do. Similarly, symbolic computation[21] of the integral in Eq. (50) gives

$$G_R(ka) = 1 - {}_1F_2\left(\frac{1}{2};\frac{3}{2},2;-k^2a^2\right). \tag{59}$$

V. NEAR-FIELD PRESSURE WHEN THE DISTANCE FROM THE CENTER OF THE DISK TO THE OBSERVATION POINT IS GREATER THAN THE DISK'S RADIUS

A. Uniform pressure distribution

Using the dipole Rayleigh integral, the sound pressure at the observation point P can be written as

$$\begin{aligned} \tilde{p}(r,\theta) = \int_{-\pi}^{\pi}\int_0^a (\tilde{p}_+(w_0) - \tilde{p}_-(w_0)) \\ \times g'(r,\theta|w_0,\phi_0)w_0dw_0d\phi_0, \end{aligned} \tag{60}$$

where g' is the normal gradient of the Green's function, as the surface of integration shrinks back to the disk, defined by

$$g'(r,\theta|w_0,\phi_0) = \frac{\partial}{\partial z_0}g(r,\theta|w_0,\phi_0,z_0)|_{z=0+}, \tag{61}$$

and $g(r,\theta|w_0,\phi_0,z_0)$ is the Green's function in cylindrical-spherical coordinates defined by $g(r,\theta|w_0,\phi_0,z_0) = e^{-ikr_1}/(4\pi r_1)$, where $r_1^2 = r^2 + w_0^2 + z_0^2 - 2r(w_0\cos\phi_0\sin\theta + z_0\cos\theta)$. The normal gradient of the Green's function is then given by

$$g'(r,\theta|w_0,\phi_0) = r\cos\theta\frac{1+ikr_0}{r_0^2}g(r,\theta|w_0,\phi_0), \tag{62}$$

where

$$g(r,\theta|w_0,\phi_0) = \frac{e^{-ikr_0}}{4\pi r_0} \tag{63}$$

and $r_0^2 = r^2 + w_0^2 - 2rw_0\cos\phi_0\sin\theta$. In order to expand g', it is first necessary to reduce it to a simpler function of g by eliminating some of the r_0 terms. This can be achieved by integrating g' with respect to θ as follows:

$$\int g'(r,\theta|w_0,\phi_0)d\theta = -\frac{g(r,\theta|w_0,\phi_0)}{w_0\cos\phi_0} \tag{64}$$

so that

$$g'(r,\theta|w_0,\phi_0) = -\frac{1}{w_0\cos\phi_0}\frac{d}{d\theta}g(r,\theta|w_0,\phi_0). \tag{65}$$

The Green's function of Eq. (63) can be expanded using the following formula, which is a special case of Gegenbauer's addition theorem:[22]

$$g(r,\theta|w_0,\phi_0) = -\frac{ik}{4\pi}\sum_{p=0}^{\infty}(2p+1)h_p^{(2)}(kr) \times j_p(kw_0)P_p(\cos\phi_0\sin\theta), \tag{66}$$

where j_p is the spherical Bessel function of the first kind and $h_p^{(2)}$ is the spherical Hankel function.[24] The Legendre function P_p can be expanded using the following addition theorem[22] (after setting one of the three angles in the original formula to $\pi/2$):

$$P_p(\cos\phi_0\sin\theta) = P_p(0)P_p(\cos\theta) + 2\sum_{q=1}^{\infty}(-1)^q P_p^{-q}(0)P_p^q(\cos\theta)\cos q\phi_0. \tag{67}$$

Inserting Eqs. (65)–(67) in Eq. (60) while noting that

$$\int_{-\pi}^{\pi}\frac{\cos q\phi_0}{\cos\phi_0}d\phi_0 = 2\pi\sin\frac{q\pi}{2}, \tag{68}$$

and applying the boundary condition of Eq. (3) leads to

$$\tilde{p}(r,\theta) = ik\tilde{p}_0\sum_{p=0}^{\infty}(2p+1)h_p^{(2)}(kr)\int_0^a j_p(kw_0)dw_0 \times \sum_{q=1}^{\infty}(-1)^q P_p^{-q}(0)\frac{d}{d\theta}P_p^q(\cos\theta)\sin\frac{q\pi}{2}. \tag{69}$$

It is noted that $P_p^{-q}(x)|_{p>q}=0$, so that the infinite limit of the summation in q can be replaced with p. Also, the even terms in p and q disappear so that

$$\tilde{p}(r,\theta) = ik\tilde{p}_0\sum_{p=0}^{\infty}(4p+3)h_{2p+1}^{(2)} \times (kr)\int_0^a j_{2p+1}(kw_0)dw_0\xi_{2p+1}(\cos\theta), \tag{70}$$

where

$$\xi_{2p+1}(\cos\theta) = \sum_{q=0}^{p}(-1)^q P_{2p+1}^{-2q-1}(0)\frac{d}{d\theta}P_{2p+1}^{2q+1}(\cos\theta). \tag{71}$$

Defining a new function

$$\chi_{2p+1}(\cos\theta) = \frac{\sqrt{\pi}(-1)^p p!}{\Gamma(p+(3/2))}\xi_{2p+1}(\cos\theta) \tag{72}$$

gives $\chi_1(\cos\theta)=\cos\theta$, $\chi_3(\cos\theta)=(5\cos^3\theta-3\cos\theta)/2$, $\chi_5(\cos\theta)=(63\cos^5\theta-70\cos^3\theta+15\cos\theta)/8$, and $\chi_7(\cos\theta)=(429\cos^7\theta-693\cos^5\theta+315\cos^3\theta-35\cos\theta)/16$. Hence it can be shown that[22]

$$\chi_{2p+1}(\cos\theta) = \frac{1}{2^{2p+1}}\sum_{q=0}^{p}\frac{(-1)^q\Gamma(4p-2q+3)(\cos\theta)^{2p-2q+1}}{q!\,\Gamma(2p-q+2)\Gamma(2p-2q+2)} = P_{2p+1}(\cos\theta), \tag{73}$$

which, after inserting Eqs. (72) and (73) in Eq. (70), gives

$$\tilde{p}(r,\theta) = \frac{ik\tilde{p}_0}{\sqrt{\pi}}\sum_{p=0}^{\infty}(-1)^p(4p+3)\frac{\Gamma(p+(3/2))}{p!} \times \int_0^a j_{2p+1}(kw_0)dw_0 h_{2p+1}^{(2)}(kr)P_{2p+1}(\cos\theta). \tag{74}$$

This is equivalent to an expression previously presented by Bouwkamp,[3] although no derivation was provided. A simpler solution[25] than Bouwkamp's to the integral over w_0 can be written as

$$\int_0^a j_{2p+1}(kw_0)dw_0 = \frac{\sqrt{\pi}}{2k}\left(\frac{ka}{2}\right)^{2p+2}\frac{{}_1F_2(p+1;p+2,2p+(5/2);-k^2a^2/4)}{(p+1)\Gamma(2p+(5/2))}, \tag{75}$$

so that, after truncating the summation limit to P, the final expression for the pressure field becomes

$$\tilde{p}(r,\theta) = -i\tilde{p}_0\sum_{p=0}^{P}\frac{(-1)^p\Gamma(p+(3/2))}{\Gamma(p+2)\Gamma(2p+(3/2))}\left(\frac{ka}{2}\right)^{2p+2} \times {}_1F_2\left(p+1;p+2,2p+\frac{5}{2};-\frac{k^2a^2}{4}\right) \times h_{2p+1}^{(2)}(kr)P_{2p+1}(\cos\theta). \tag{76}$$

This expansion converges providing $r\geqslant a$, and was used for the region $w\geqslant a$. It is similar in form to the "outer" solution obtained by Mast and Yu[16] for the piston in an infinite baffle, except that the current solution is an expansion of the odd terms of the spherical Hankel and Legendre functions as opposed to even ones. These odd eigenfunctions are a result of the odd boundary condition given by Eq. (1). Let an error function be defined by

$$\varepsilon(r,\theta) = \frac{|\tilde{p}(r,\theta)-\tilde{p}_{ref}(r,\theta)|}{|\tilde{p}(r,\theta)|}, \tag{77}$$

where the reference pressure is that obtained using the original dipole Rayleigh integral or Eq. (60) with the unexpanded Green's function normal gradient of Eq. (62). The calculations were performed using 30 digit precision with $P\approx 4ka$, whereby P was rounded down to the nearest integer value. This produced values of ε typically less than 0.00001, but rising to around 0.1 in the immediate vicinity of the rim ($r=a$ and $\theta=\pi/2$) where convergence was slowest. At $ka=6\pi$, the expansion calculated four times faster than numerical integration. Furthermore, the calculation time for the expansion roughly doubles for every doubling of ka, whereas for numerical integration it nearly quadruples for every doubling of ka.

B. Generalization to an arbitrary surface pressure distribution

Let the arbitrary surface pressure distribution be defined by the power series

$$\tilde{p}_+(w_0) = -\tilde{p}_-(w_0) = \sum_{m=0}^{M} \tilde{A}_m \left(1 - \frac{w_0^2}{a^2}\right)^{m+1/2}, \quad 0 \leq w_0 \leq a, \tag{78}$$

where, in the case of a rigid disk in free space,[7] the unknown series coefficients $\tilde{A}_m$ are related to normalized coefficients $\tilde{\tau}_m$ by $\tilde{A}_m = \tilde{\tau}_m (m + (3/2)) ka\rho c\tilde{u}_0$, or, in the case of a membrane in free space,[11] by $\tilde{A}_m = \tilde{\tau}_m (m + (3/2))\tilde{p}_I/2$. Inserting the above expressions in Eq. (60) and using the identity[21]

$$\int_0^a \left(1 - \frac{w_0^2}{a^2}\right)^{m+1/2} j_{2p+1}(kw_0)dw_0 = \frac{\sqrt{\pi}}{2k} \frac{\Gamma(m+(3/2))p!}{\Gamma(p+m+(5/2))\Gamma(2p+(5/2))} \left(\frac{ka}{2}\right)^{2p+2} \times {}_1F_2\left(p+1; p+m+\frac{5}{2}, 2p+\frac{5}{2}; -\frac{k^2a^2}{4}\right), \tag{79}$$

leads to the final expression for the pressure field which is given by

$$\tilde{p}(r,\theta) = 2i \sum_{m=0}^{M} \tau_m \sum_{p=0}^{P} \frac{(-1)^p \Gamma(p+(3/2)) h_{2p+1}^{(2)}(kr) P_{2p+1}(\cos\theta)}{\Gamma(2p+(3/2))(m+(5/2))_p} \times \left(\frac{ka}{2}\right)^{2p+2} {}_1F_2\left(p+1; p+m+\frac{5}{2}, 2p+\frac{5}{2}; -\frac{k^2a^2}{4}\right) \times \begin{cases} ka\rho c\tilde{u}_0, & \text{Rigid disk} \\ \tilde{p}_I/2, & \text{Membrane} \end{cases}. \tag{80}$$

where $\tilde{u}_0$ is the disk velocity and $\tilde{p}_I$ is the membrane driving pressure.

VI. NEAR-FIELD PRESSURE PARAXIAL SOLUTION

In order to find a solution which converges up to the face of the disk, a trick previously used by Mast and Yu[16] is to move the center of the coordinate system from the center of the disk out to the same axial distance as the observation point. Referring to Fig. 1 the distance r_0 from a point source on the disk to the observation point is $r_0^2 = r_1^2 + w^2 - 2r_1 w \cos\phi_0 \sin\beta$, where $w = r\sin\theta$, $r_1 = \sqrt{z^2 + w_0^2}$, and $z = r\cos\theta$. The angle β is defined by $\cos\beta = z/r_1$ and $\sin\beta = w_0/r_1$. Putting these new parameters in Eq. (74) gives

$$\tilde{p}(w,z) = \frac{ik\tilde{p}_0}{\sqrt{\pi}} \sum_{p=0}^{\infty} (-1)^p (4p+3) \frac{\Gamma(p+(3/2))}{\Gamma(p+1)} \times \frac{j_{2p+1}(kw)}{w} \int_z^{r_a} h_{2p+1}^{(2)}(kr_1) P_{2p+1}(\cos\beta) r_1 dr_1, \tag{81}$$

where $r_a = \sqrt{z^2 + a^2}$. Let

$$\tilde{p}(w,z) = \frac{i\tilde{p}_0}{\sqrt{\pi kw}} \sum_{p=0}^{P} (-1)^p (4p+3) \times \frac{\Gamma(p+(3/2))}{\Gamma(p+1)} j_{2p+1}(kw) f_{2p+1}, \tag{82}$$

where, after substituting $\zeta = kr_1$,

$$f_{2p+1} = \int_{kz}^{kr_a} h_{2p+1}^{(2)}(\zeta) P_{2p+1}\left(\frac{kz}{\zeta}\right) \zeta d\zeta. \tag{83}$$

Then the following indefinite integral g_{2p+1} is denoted by

$$g_{2p+1}(\zeta) = \int h_{2p+1}^{(2)}(\zeta) P_{2p+1}\left(\frac{kz}{\zeta}\right) \zeta d\zeta, \tag{84}$$

so that $f_{2p+1} = g_{2p+1}(kr_a) - g_{2p+1}(kz)$. When $p=0$, the first term is given by

$$g_1(\zeta) = \int h_1^{(2)}(\zeta) P_1\left(\frac{kz}{\zeta}\right) \zeta d\zeta = i\frac{kz}{\zeta} e^{-i\zeta}, \tag{85}$$

so that

$$f_1 = i\left(\frac{z}{r_a} e^{-ikr_a} - e^{-ikz}\right). \tag{86}$$

According to Hasagawa *et al.*,[15] the remaining terms can be determined from the following recursion formulas:

$$g_{2p+1}(\zeta) + g_{2p-1}(\zeta) = \zeta h_{2p}^{(2)}(\zeta)\left(P_{2p+1}\left(\frac{kz}{\zeta}\right) - P_{2p-1}\left(\frac{kz}{\zeta}\right)\right), \tag{87}$$

$$g_{2p+1}(kz) + g_{2p-1}(kz) = 0. \tag{88}$$

Hence

$$f_{2p+1} = -f_{2p-1} + kr_a h_{2p}^{(2)}(kr_a)\left(P_{2p+1}\left(\frac{z}{r_a}\right) - P_{2p-1}\left(\frac{z}{r_a}\right)\right). \tag{89}$$

Thus the solution is given by the combination of Eqs. (82), (86), and (89), which converges for $w^2 < a^2 + z^2$. Again, this is essentially an odd term version of the "paraxial" expansion obtained by Mast and Yu[16] for the piston in an infinite baffle. Toward the axis of symmetry, convergence is achieved with fewer terms until only the first term of the expansion remains, which is the closed-form axial solution. This can also be derived directly by setting $\theta=0$ in Eq. (63) before integrating over the surface to give

$$\tilde{p}(r,0) = \frac{\tilde{p}_0}{2}\left(e^{-ikr} - \frac{r}{\sqrt{r^2+a^2}} e^{-ikr\sqrt{r^2+a^2}}\right). \tag{90}$$

For comparison, the Backhaus and Trendelenburg axial solution[26] for a piston in an infinite baffle is

$$\tilde{p}(r,0) = \rho c\tilde{u}_0 (e^{-ikr} - e^{-ik\sqrt{r^2+a^2}}). \tag{91}$$

The calculations for Eqs. (82), (86), and (89) were performed using 30 digit precision for the region $w < a$ with $P \approx 4kw$, whereby P was rounded down to the nearest integer value.

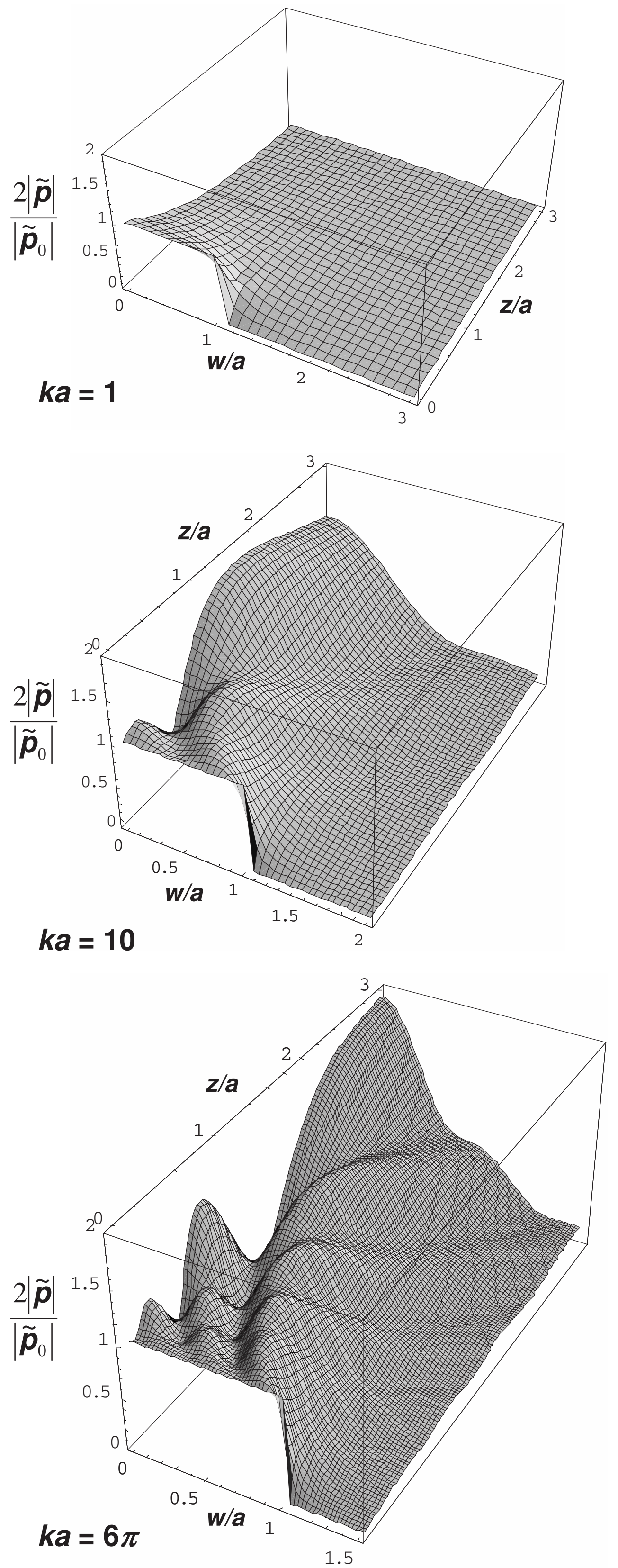

FIG. 4. Near-field pressure of the resilient disk in free space for $ka=1$, 10, and 6π.

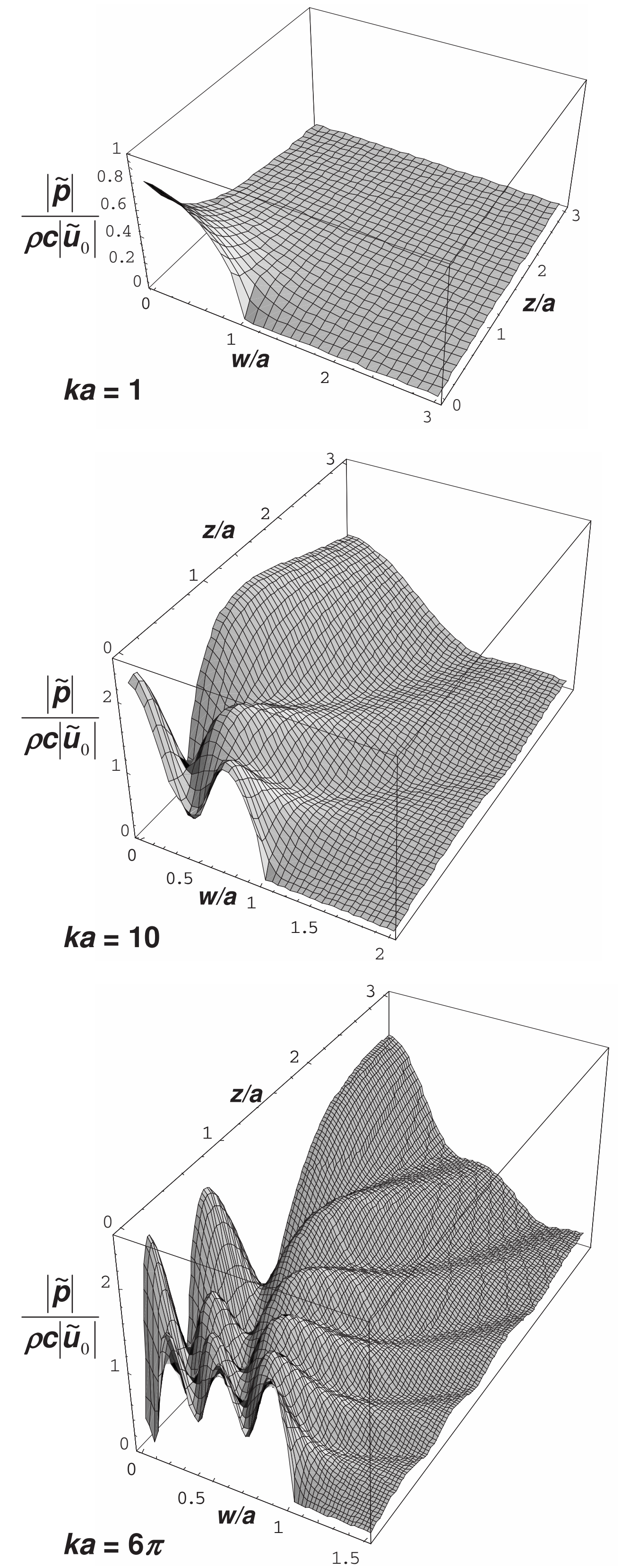

FIG. 5. Near-field pressure of the rigid disk in free space for $ka=1$, 10, and 6π.

Again, this produced values of ε typically of the order of 0.00001, but rising to around 0.1 in the immediate vicinity of the rim. For various values of ka, pressure fields are shown in Fig. 4 for a resilient disk in free space and in Fig. 5 for a rigid disk in free space. The latter were calculated using Eq. (80) for $r \geqslant a$ and, in the case of $r < a$, using similar formulation to that previously derived for a membrane in free space.[11]

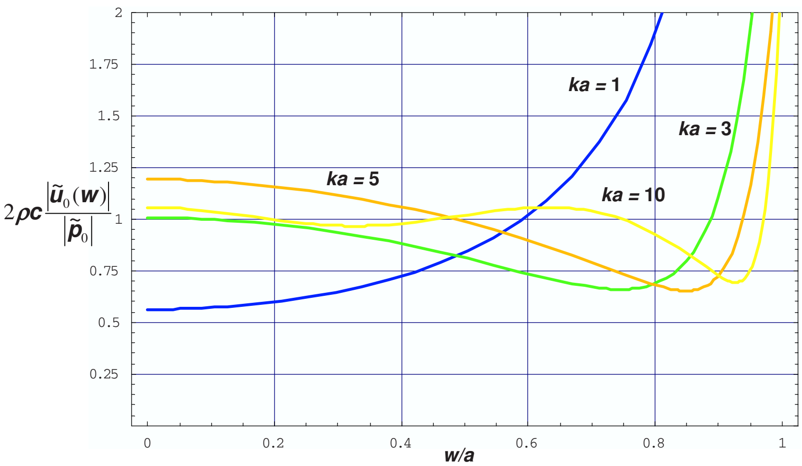

FIG. 6. (Color online) Normalized surface velocity magnitude of the resilient disk in free space.

VII. SURFACE VELOCITY

Using the solutions for the near-field pressure from Eqs. (82), (86), and (89), the surface velocity is given by

$$\tilde{u}_0(w) = \frac{i}{k\rho c}\frac{d}{dz}\tilde{p}(w,z)\bigg|_{z=0+}$$
$$= -\frac{\tilde{p}_0}{\rho c\sqrt{\pi}}\sum_{p=0}^{\infty}(-1)^p(4p+3)\times\frac{\Gamma(p+(3/2))}{\Gamma(p+1)}f'_{2p}\frac{j_{2p+1}(kw)}{kw}, \quad (92)$$

where

$$f'_0 = 1 - i\frac{e^{-ika}}{ka}, \quad (93)$$

$$f'_{2p} = -f'_{2p-2} - h^{(2)}_{2p}(ka)\times((2p+1)P_{2p}(0)-(2p-1)P_{2p-2}(0)). \quad (94)$$

The magnitude and phase of the normalized velocity are shown in Figs. 6 and 7, respectively, for various values of ka. For small k, it can be shown to agree well with asymptotic expression given by Eq. (10).

VIII. FAR-FIELD PRESSURE

A far-field expression can be derived using the usual procedure,[7,20] which involves an asymptotic version of Eq. (62) for large r, giving

$$\tilde{p}(r,\theta) = -\frac{ia\tilde{p}_0}{4r}e^{-ikr}D(\theta), \quad (95)$$

where $D(\theta)$ is the directivity function, which is given by

$$D(\theta) = \frac{2J_1(ka\sin\theta)}{\sin\theta}\cos\theta \quad (96)$$

and plotted in Fig. 8 for various values of ka. The directivity function of a rigid disk in free space is plotted in Fig. 9 for comparison. Since $D(0)=ka$, the on-axis response is simply

$$\tilde{p}(r,0) = -\frac{ika^2\tilde{p}_0}{4r}e^{-ikr}, \quad (97)$$

which just gives a constant 6 dB/octave rising slope at all frequencies. In the case of an electrostatic loudspeaker

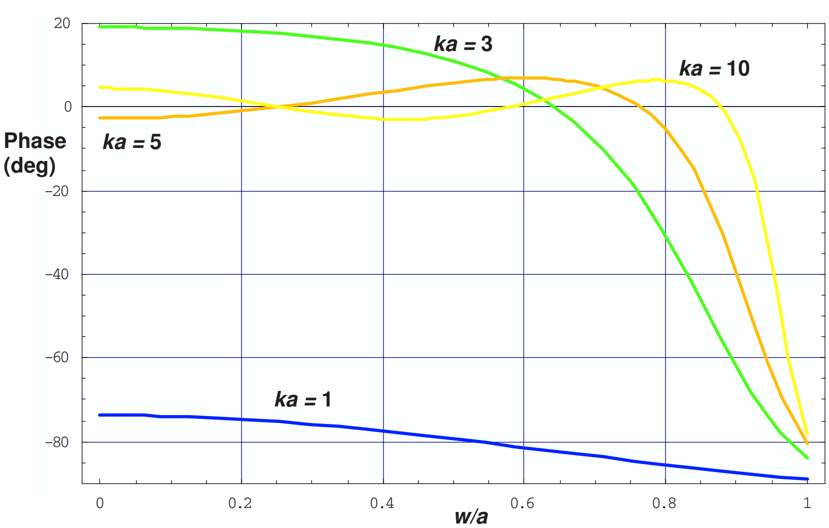

FIG. 7. (Color online) Surface velocity phase of the resilient disk in free space.

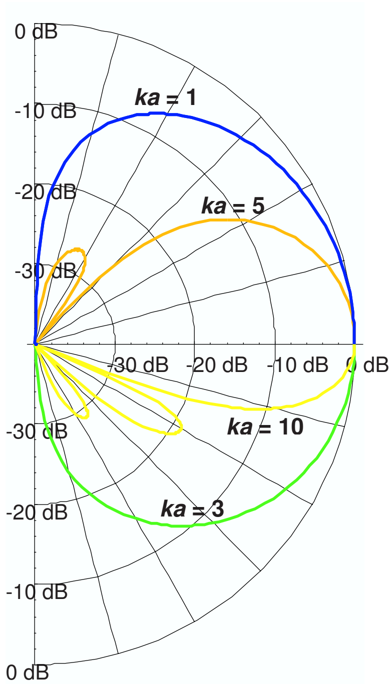

FIG. 8. (Color online) Normalized far-field directivity function of the resilient disk in free space.

$$\tilde{p}_0 = \frac{E_P}{d} \cdot \frac{2\tilde{I}_{in}}{i\omega\pi a^2}, \tag{98}$$

where E_P is the polarizing voltage, d is the membrane-electrode separation, and $\tilde{I}_{in}$ is the input static current, assuming that the motional current is negligible in comparison. Substituting this in Eq. (97) yields

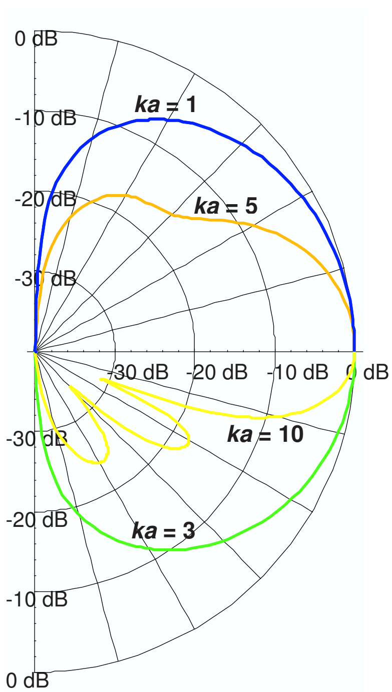

FIG. 9. (Color online) Normalized far-field directivity function of the rigid disk in free space.

$$\tilde{p}(r,0) = -\frac{E_P}{d} \cdot \frac{\tilde{I}_{in}}{2\pi rc} e^{-ikr}, \tag{99}$$

which is Walker's equation,[18] albeit obtained by a slightly different method.

IX. DISCUSSION

The results presented here are intended to provide a full set of radiation characteristics for a fundamental axisymmetric planar dipole source. Starting with the radiation admittance and impedance, it can be seen from Figs. 2 and 3 that in the case of unbaffled radiators, the rigid disk has ripples in both the real and imaginary parts, whereas those of the resilient disk are smooth almost monotonic functions, as are also those of the oscillating rigid sphere (or Morse Ingard approximation). These are entirely consistent with the results previously found for baffled radiators.[5] The solution presented for the susceptance integral contradicts a previous statement[7] made by the author that it could not be solved. At least, that is how it seemed when trying to tackle it numerically. The integrand is strongly oscillatory and converges more slowly than just about any other integral in fundamental sound radiation theory, due to the combination of the singularity and acoustic short circuit at the rim. Also, an attempt to solve it by symbolic computation gave somewhat unexpected results as discussed in Sec. IV E.

At medium to high frequencies, the pressure field can be divided into two regions: the Fresnel diffraction region in the near field and the Fraunhofer diffraction region in the far field. It is well understood that the Fresnel region is characterized by complex non-propagating interference patterns, whereas the Fraunhofer region is characterized by spherically propagating waves in a beam pattern with a strong central lobe (Airy disk) accompanied by smaller side lobes (Airy pattern). At higher frequencies, a third "shadow region" begins to form at the surface of the radiator which is characterized by plane waves in a straight beam with the same cross section as the radiator, like a virtual transmission line in space. It appears from Figs. 4 and 5 that this shadow region forms more readily at lower frequencies in the case of the resilient disk, no doubt aided by the constant pressure distribution at the surface of the disk. At $ka=6\pi$, the pressure field fluctuations in the vicinity of the rigid disk are considerably greater than for the resilient disk. Furthermore, the axial pressure response of a rigid disk given by Eq. (91) has nulls, whereas the resilient disk axial response given by Eq. (90) is oscilliatory but with decreasing magnitude towards the face of the disk.

In Figs. 6 and 7, the velocity magnitude and phase distributions are generally similar in shape to those of the resilient disk in an infinite baffle, except that the magnitude rises more rapidly towards the rim due to the acoustic short circuit there in the absence of a baffle. This is also accompanied by increased phase shift.

Previously, the author has stated that at high frequencies the directivity function of the resilient disk in free space does not converge towards that of a rigid disk in an infinite baffle.[6] This is not strictly true. They must converge eventually because the only difference between them is a factor of cos θ, which produces the classic figure-of-eight dipole pattern at low frequencies. As the directivity pattern becomes narrower, this factor has less of an effect. However, the convergence is rather slow. At $ka=10$, the beam pattern of an unbaffled rigid disk, shown in Fig. 9, is rather similar to that of a baffled one, except that the latter has nulls due to the *velocity* boundary conditions on both the disk and surrounding baffle. This can also be derived by taking the Hankel transform of the velocity distribution in the plane of the disk, which is a rotationally symmetric step function. By contrast, the side lobes of the unbaffled resilient disk, shown in Fig. 8, are somewhat smaller. This pattern also has nulls due to the *pressure* boundary conditions on both the disk and surrounding plane.

X. CONCLUSIONS

Closed-form expressions have been derived for the radiation admittance of a resilient disk in free space and these have been shown to have a simple relationship with the radiation impedance of a rigid piston in an infinite baffle. Also, rapidly converging single expansions have been derived for the near-field pressure and surface velocity. Furthermore, the near-field pressure has been generalized to an arbitrary surface pressure distribution, yielding a double expansion, whereas a triple expansion was previously derived by the author for a membrane in free space.[11] Hence the solution is now similar in form to that of the membrane in an infinite baffle.[11]

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