

On the sound field of a resilient disk in an infinite baffle

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The Rayleigh integral describing the near-field pressure of an axisymmetric planar monopole source with an arbitrary velocity distribution is solved with a method similar to that used by Mast and Yu [J. Acoust. Soc. Am. **118**(6), 3457–3464 (2005)] for a rigid disk in an infinite baffle. The closed-form solution is in the form of a double expansion, which is valid for distances from the observation point to the center of the source that are greater than its radius. However, for the remaining immediate near field, the King integral is solved using a combination of Gegenbauer's summation theorem and the Lommel expansion, resulting in a solution which is in the form of a triple expansion, reducing to a double expansion along the source's axis of symmetry. These relatively compact solutions in analytic form do not require numerical integration and therefore present no numerical difficulties except for a singularity at the rim. As an example of a monopole source with an arbitrary velocity distribution, equations describing the radiation characteristics of a resilient disk in an infinite baffle are derived. Using Babinet's principle, the pressure field of a plane wave passing through the complementary hole in an infinite rigid screen is calculated.

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I. INTRODUCTION

Despite the advances in acoustical simulation tools in recent years, it is still useful to have some canonical forms which can be rigorously calculated in a direct manner without any need for numerical integration, iteration, or least-squares minimization. Although such solutions are generally restricted to simple geometries (usually axisymmetric), they provide useful benchmarks for simulation. Most canonical forms, such as the spherical polar cap on a sphere,¹ are based upon spherical geometry.

Recently, Mast and Yu² derived an elegant pair of expansions for the pressure field of a rigid disk in an infinite baffle, based upon solutions to the Rayleigh³ integral. The first expansion, or “outer” solution, provides a fast converging series for distances from the center of the disk to the observation point greater than the disk's radius, and can be regarded as an improved version of Stenzel's solution⁴ using functions commonly found in text books as opposed to Stenzel's bespoke ones, which were based upon Lommel's polynomials.

Stenzel also derived what could be termed an “inner” solution for distances from the center of the disk to the observation point less than the disk's radius. This expansion can also be updated using hypergeometric function solutions to the radial integrals, which the author has already tried. However, although it converges in a similar fashion to the outer expansion, this is not the best solution available. In all of the aforementioned solutions, the origin of the spherical coordinate system is located at the center of the disk.

Hasegawa *et al.*⁵ provided an expansion, together with recursion formulas, based upon an ingenious spherical coordinate system, the origin of which being located on the disk's axis of symmetry some distance in front of it. Mast and Yu² have developed this by locking the origin of the coordinate system to the same axial distance from the disk as the obser-

vation point, while keeping it on the axis of symmetry. This leads to their second expansion or “paraxial” solution for which the region of convergence looks like a funnel, falling just within the disk's perimeter at its surface and then spreading out to form a cone covering an angle of 45° either side of the axis of symmetry with the apex located at the disk's center. The limit of this expansion decreases to a single term on the axis of symmetry.

Hence, the Rayleigh integral has been shown to provide simple single expansion solutions for a rigid disk. In this paper, it is also shown to provide a similar outer solution for a monopole source with an arbitrary velocity distribution. However, for an inner or paraxial solution this does not appear to be the case and Stenzel⁶ tackled the problem in a somewhat formidable analysis.

Surprisingly little attention has been paid to the King integral,^{7–9} which could be solved by means of a double expansion. Greenspan¹⁰ calculated it numerically to illustrate some special cases, such as the on-axis pressure, radiation force, and power for various monopole velocity distributions, as well as the transient response. Using two Lommel expansions, Williams¹¹ recast the King integral in a mathematically beautiful form, which he then used to illustrate some of its properties. Unfortunately, it can be shown that the integral in this expanded form yields a converging solution for only part of the near-field space. When the reverse Lommel expansions are applied, it is not obvious how to calculate the subsequent expression numerically and no results are provided.

In the current paper, the King integral is expanded using a combination of the Lommel expansion and the Gegenbauer summation theorem in order to ensure convergence. This yields a paraxial solution, which together with the outer solution, exhibits similar convergence characteristics to the expansions of Mast and Yu for the rigid disk.

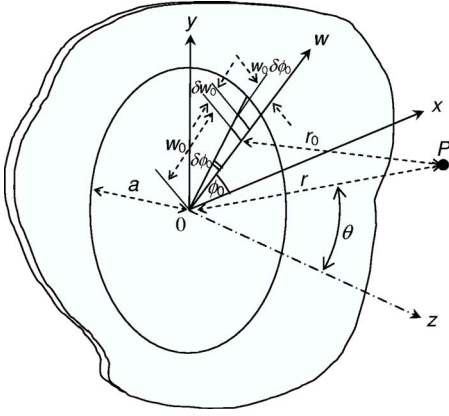


FIG. 1. Geometry of the disk in an infinite baffle.

A general aim of this paper is to present the most direct solutions possible for calculating the sound radiation characteristics of axisymmetric *monopole* sources, especially when the *velocity* distribution is unknown. In this respect, it is intended to complement a recent paper by Kärkkäinen *et al.*,¹² in which a solution was presented for calculating the radiation characteristics of *dipole* sources with unknown *pressure* distributions. The surface pressure distribution was described using a trial function consisting of a power series for which the coefficients were calculated by numerically solving a set of simultaneous equations. The simultaneous equations, in turn, had to be derived by analytically solving the integrals in the dipole part of the Kirchhoff-Helmholtz boundary integral formula (using the Green's function in cylindrical coordinates). In this paper, a similar procedure is followed, using a trial function for the surface velocity distribution and then solving the monopole part of the Kirchhoff-Helmholtz boundary integral formula.

Furthermore, the methods derived here can also be applied to nonrigid sources with fluid-structure coupling, such as plates and membranes. For example, similar formulation appears in a study on membranes by Kärkkäinen *et al.*¹³

II. RESILIENT DISK IN AN INFINITE BAFFLE

A. Boundary conditions

The infinitesimally thin rigid disk shown in Fig. 1 lies in the w plane with its center at the origin, where w is the radial ordinate of the cylindrical coordinate system and z is the axial ordinate. Due to axial symmetry, the polar ordinate ϕ of the spherical coordinate system can be ignored, so that the observation point P is defined simply in terms of the radial and azimuthal ordinates r and θ , respectively. The infinitesimally thin rigid baffle extends from the perimeter of the disk to $w=\infty$. On the baffle, the velocity is zero and therefore so is the normal pressure gradient. The infinitesimally thin membrane-like resilient disk is assumed to be perfectly flexible, has zero mass, and is free at its perimeter. It is driven by a uniformly distributed harmonically varying pressure \tilde{p}_0 and thus radiates sound from both sides into a homogeneous loss-free acoustic medium. In fact, there need not be a disk present at all and instead the driving pressure could be acting upon the air particles directly. However, for expedience, the

area over which this driving pressure is applied shall be referred to as a disk from here onwards. On the surface of the disk and baffle, the following boundary conditions apply:

$$\frac{\partial}{\partial z_0} \tilde{p}(w_0, z_0)|_{z_0=0+} = \begin{cases} -ik\rho c \tilde{u}(w_0), & 0 \leq w_0 \leq a, \\ 0, & w_0 > a, \end{cases} \quad (1)$$

where

$$\tilde{u}_0(w_0) = \sum_{m=0}^{\infty} \tilde{A}_m \left(1 - \frac{w_0^2}{a^2}\right)^{m-(1/2)}, \quad (2)$$

and k is the wave number given by

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}, \quad (3)$$

where ω is the angular frequency of excitation, ρ is the density of the surrounding medium, c is the speed of sound in that medium, and λ is the wavelength. The annotation $\tilde{}$ denotes a harmonically time-varying quantity and replaces the factor $e^{i\omega t}$. It is worth noting that the index of the first term of the expansion ($m=0$) is equal to $-1/2$, in order to satisfy the boundary condition of infinite velocity at the perimeter, as determined by Rayleigh.³ The same expansion can be applied to any velocity distribution, providing the velocity is either infinite or zero at the perimeter. For example, in the case of a circular membrane with a clamped rim, the index of the first term would be equal to $+1/2$.

On the front and rear surfaces of the disk, the pressures are \tilde{p}_+ and \tilde{p}_- , respectively, which are given by

$$\tilde{p}_+ = -\tilde{p}_- = \frac{\tilde{p}_0}{2}. \quad (4)$$

B. Solution of the free-space wave equation

Using the King integral, the pressure distribution is defined by

$$\tilde{p}(w, z) = 2 \int_0^{2\pi} \int_0^a g(w, z|w_0, z_0) \times \frac{\partial}{\partial z_0} \tilde{p}(w_0, z_0)|_{z_0=0+} w_0 dw_0 d\phi_0, \quad (5)$$

where the Green's function¹⁴ is defined, in cylindrical coordinates, by the Lamb¹⁵ or Sommerfeld¹⁶ integral,

$$g(w, z|w_0, z_0) = \frac{i}{4\pi} \int_0^{\infty} J_0(\mu w) J_0(\mu w_0) \frac{\mu}{\sigma} e^{-i\sigma|z-z_0|} d\mu, \quad (6)$$

where

$$\sigma = \sqrt{k^2 - \mu^2}. \quad (7)$$

Substituting Eqs. (1), (2), and (6), in Eq. (5) and integrating over the surface of the disk yields

$$\begin{aligned} \tilde{p}(w, z) &= ka\rho c \sum_{m=0}^{\infty} \tilde{A}_m \Gamma\left(m + \frac{1}{2}\right) \int_0^{\infty} \left(\frac{2}{a\mu}\right)^{m-(1/2)} \\ &\quad \times J_0(w\mu) J_{m+(1/2)}(a\mu) \frac{e^{-i\sigma z}}{\sigma} d\mu, \end{aligned} \quad (8)$$

where Sonine's integral¹⁷ has been used as follows:

$$\begin{aligned} \int_0^a \left(1 - \frac{w_0^2}{a^2}\right)^{m-(1/2)} J_0(\mu w_0) w_0 dw_0 &= \frac{a^2}{2} \Gamma\left(m + \frac{1}{2}\right) \\ &\quad \times \left(\frac{2}{a\mu}\right)^{m+(1/2)} J_{m+(1/2)}(a\mu). \end{aligned} \quad (9)$$

Applying the boundary condition of Eq. (4) leads to

$$\begin{aligned} \frac{\tilde{p}_0}{2} &= ka\rho c \sum_{m=0}^{\infty} \tilde{A}_m \Gamma\left(m + \frac{1}{2}\right) \int_0^{\infty} \left(\frac{2}{a\mu}\right)^{m-(1/2)} \\ &\quad \times J_0(w\mu) J_{m+(1/2)}(a\mu) \frac{1}{\sigma} d\mu. \end{aligned} \quad (10)$$

C. Formulation of the coupled problem

Equation (10) can be written more simply as

$$\sum_{m=0}^{\infty} \tau_m I_m(w, k) = 1, \quad (11)$$

which is to be solved for the normalized power series coefficients τ_m as defined by

$$\tau_m = \frac{2\rho c \tilde{A}_m}{(m + 1/2) \tilde{p}_0}. \quad (12)$$

The integral $I_m(w, k)$ can be split into two parts,

$$I_m(w, k) = I_{mR}(w, k) - iI_{mI}(w, k), \quad (13)$$

where the real part is given by

$$\begin{aligned} I_{mR}(w, k) &= ka\Gamma\left(m + \frac{3}{2}\right) \int_0^k \left(\frac{2}{\mu a}\right)^{m-(1/2)} J_{m+(1/2)} \\ &\quad \times (\mu a) J_0(\mu w) \frac{1}{\sqrt{k^2 - \mu^2}} d\mu, \end{aligned} \quad (14)$$

and the imaginary part is given by

$$\begin{aligned} I_{mI}(w, k) &= ka\Gamma\left(m + \frac{3}{2}\right) \int_k^{\infty} \left(\frac{2}{\mu a}\right)^{m-(1/2)} J_{m+(1/2)} \\ &\quad \times (\mu a) J_0(\mu w) \frac{1}{\sqrt{\mu^2 - k^2}} d\mu. \end{aligned} \quad (15)$$

D. Solution of the real integral

Substitution of $\mu = k \sin \vartheta$ in Eq. (14) gives

$$\begin{aligned} I_{mR}(w, k) &= ka\Gamma\left(m + \frac{3}{2}\right) \left(\frac{2}{ka}\right)^{m-(1/2)} \int_0^{\pi/2} (\sin \vartheta)^{(1/2)-m} \\ &\quad \times J_{m+(1/2)}(ka \sin \vartheta) J_0(kw \sin \vartheta) d\vartheta. \end{aligned} \quad (16)$$

The Bessel functions in Eq. (16) are defined by¹⁷

$$J_0(kw \sin \vartheta) = \sum_{q=0}^Q \left(\frac{kw}{2}\right)^{2q} \frac{(-1)^q (\sin \vartheta)^{2q}}{(q!)^2}, \quad (17)$$

$$\begin{aligned} J_{m+(1/2)}(ka \sin \vartheta) &= \sum_{r=0}^R \left(\frac{ka}{2}\right)^{2r+m+(1/2)} \\ &\quad \times \frac{(-1)^r (\sin \vartheta)^{2r+m+(1/2)}}{r! \Gamma(r + m + 3/2)}, \end{aligned} \quad (18)$$

so that

$$\begin{aligned} I_{mR}(w, k) &= 2 \sum_{q=0}^Q \sum_{r=0}^R \frac{(-1)^{q+r} \Gamma(m + 3/2)}{(q!)^2 r! \Gamma(r + m + 3/2)} \left(\frac{ka}{2}\right)^{2(q+r+1)} \\ &\quad \times \left(\frac{w}{a}\right)^{2q} \int_0^{\pi/2} (\sin \vartheta)^{2(q+r)+1} d\vartheta. \end{aligned} \quad (19)$$

Solution of the integral in Eq. (19) is enabled by use of the following identity:¹⁸

$$\int_0^{\pi/2} (\sin \vartheta)^{2(q+r)+1} d\vartheta = \frac{\sqrt{\pi} \Gamma(q + r + 1)}{2 \Gamma(q + r + 3/2)}. \quad (20)$$

Evaluating the integral over ϑ yields

$$\begin{aligned} I_{mR}(w, k) &= \sqrt{\pi} \sum_{q=0}^Q \sum_{r=0}^R \frac{(-1)^{q+r} \Gamma(m + 3/2) \Gamma(q + r + 1)}{(q!)^2 r! \Gamma(r + m + 3/2) \Gamma(q + r + 3/2)} \\ &\quad \times \left(\frac{ka}{2}\right)^{2(q+r+1)} \left(\frac{w}{a}\right)^{2q}. \end{aligned} \quad (21)$$

E. Solution of the imaginary integral

1. Transformation of the integral into complex form

The following procedure converts the infinite limit of the integral in Eq. (15) into a finite one. First, the integral is converted into a form which can be integrated in the complex plane. The Bessel function $J_n(x)$ can be written in terms of the following pair of complex conjugate Hankel functions:¹⁷

$$J_n(x) = \frac{H_n^{(1)}(x) + H_n^{(2)}(x)}{2}, \quad (22)$$

which can now be used to separate I_{mI} into two complex conjugate integrals as follows:

$$I_{mI}(w, k) = \frac{I_{mI}^{(1)} + I_{mI}^{(2)}}{2}, \quad (23)$$

where, after substituting $\mu = kt$, the complex conjugate integrals are given by

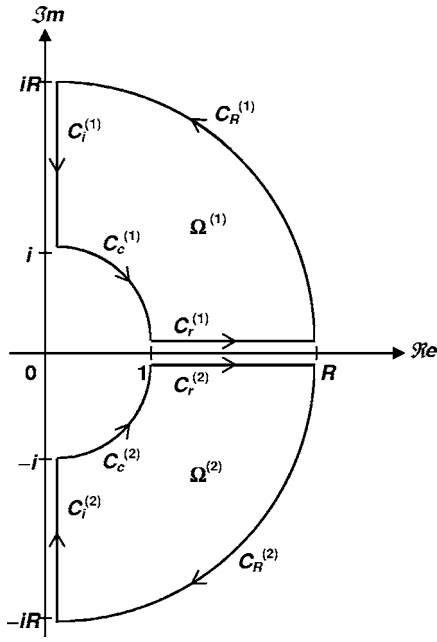


FIG. 2. Infinite integration contours in the complex t plane.

$$I_{ml}^{(1)} = 2\Gamma\left(m + \frac{3}{2}\right)\left(\frac{2}{ka}\right)^{m-(3/2)} \int_1^\infty J_0(kwt)H_{m+(1/2)}^{(1)}(kat) \frac{t^{(1/2)-m}}{\sqrt{t^2-1}} dt, \quad (24)$$

$$I_{ml}^{(2)} = 2\Gamma\left(m + \frac{3}{2}\right)\left(\frac{2}{ka}\right)^{m-(3/2)} \int_1^\infty J_0(kwt)H_{m+(1/2)}^{(2)}(kat) \frac{t^{(1/2)-m}}{\sqrt{t^2-1}} dt. \quad (25)$$

Referring to the complex t plane of Fig. 2, the integrals $I_{ml}^{(1)}$ and $I_{ml}^{(2)}$ can now be evaluated along contours $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively. The contours are defined by

$$\begin{aligned} \Omega^{(1)} &= t \in (C_r^{(1)} \cup C_R^{(1)} \cup C_i^{(1)} \cup C_c^{(1)}), \\ C_r^{(1)} &= [1, R], \\ C_R^{(1)} &= [Re^{i\vartheta} | 0 \leq \vartheta \leq \pi/2], \\ C_i^{(1)} &= [iR, i], \\ C_c^{(1)} &= [e^{i\vartheta} | \pi/2 \geq \vartheta \geq 0], \\ R &\rightarrow \infty, \\ \Omega^{(2)} &\text{ symmetric to } \Omega^{(1)} \text{ with respect to the real axis.} \end{aligned} \quad (26)$$

2. Contribution of $C_R^{(1)}$ and $C_R^{(2)}$

The contributions along $C_R^{(1)}$ and $C_R^{(2)}$ vanish for $R \rightarrow \infty$ due to the behavior of $H_{m+(1/2)}^{(1)}(t)$ as $|t| \rightarrow \infty$.

3. Contribution of $C_i^{(1)}$ and $C_i^{(2)}$

Noting that $\sqrt{t^2-1} = i\sqrt{1-t^2}$, the integral along $C_i^{(1)}$ can be written

$$2\Gamma\left(m + \frac{3}{2}\right)\left(\frac{2}{ka}\right)^{m-(3/2)} \int_{i\infty}^i J_0(kwt)H_{m+(1/2)}^{(1)}(kat) \frac{t^{(1/2)-m}}{i\sqrt{1-t^2}} dt, \quad (27)$$

which can be converted into an integral with real limits by substituting $t=is$ as follows:

$$-2i^{(1/2)-m}\Gamma\left(m + \frac{3}{2}\right)\left(\frac{2}{ka}\right)^{m-(3/2)} \int_1^\infty J_0(ikws)H_{m+(1/2)}^{(1)}(kas) \frac{s^{(1/2)-m}}{\sqrt{1+s^2}} ds. \quad (28)$$

With help from the following identities:¹⁷

$$I_0(x) = J_0(ix), \quad (29)$$

$$K_\nu(x) = i^{\nu+1} \frac{\pi}{2} H_\nu^{(1)}(ix), \quad (30)$$

the integral can be written as

$$\frac{4i}{\pi} (-1)^m \Gamma\left(m + \frac{3}{2}\right)\left(\frac{2}{ka}\right)^{m-(3/2)} \int_1^\infty I_0(kws)K_{m+(1/2)}(kas) \frac{s^{(1/2)-m}}{\sqrt{1+s^2}} ds. \quad (31)$$

This integral is purely imaginary whereas the original is real valued. Hence, there is zero net contribution along $C_i^{(1)}$. The same is true for the contribution along $C_i^{(2)}$ where $\sqrt{t^2-1} = -i\sqrt{1-t^2}$.

4. Contribution of $C_c^{(1)}$ and $C_c^{(2)}$

Finally, the contributions along the unity quarter circle segments $C_c^{(1)}$ and $C_c^{(2)}$ can be calculated by using the substitution $t=e^{i\vartheta}$, so that the contribution along $C_c^{(1)}$ becomes

$$\Re\left(2i\Gamma\left(m + \frac{3}{2}\right)\left(\frac{2}{ka}\right)^{m-(3/2)} \int_{\pi/2}^0 J_0(kwe^{i\vartheta})H_{m+(1/2)}^{(1)}(kae^{i\vartheta}) \frac{e^{i((3/2)-m)\vartheta}}{\sqrt{e^{2i\vartheta}-1}} d\vartheta\right), \quad (32)$$

and likewise the contribution along $C_c^{(2)}$ becomes

$$\Re\left(2i\Gamma\left(m + \frac{3}{2}\right)\left(\frac{2}{ka}\right)^{m-(3/2)} \int_{-\pi/2}^0 J_0(kwe^{i\vartheta})H_{m+(1/2)}^{(2)}(kae^{i\vartheta}) \frac{e^{i((3/2)-m)\vartheta}}{\sqrt{e^{2i\vartheta}-1}} d\vartheta\right), \quad (33)$$

which is equal to Eq. (32). As there are no poles or zeros within the contours $\Omega^{(1)}$ or $\Omega^{(2)}$, it can be stated that, according to the residue theorem, the sum of the integrals around each of these contours is equal to zero. Therefore, I_{ml} can be written as

$$\begin{aligned}
I_{ml}(w,k) &= \frac{I_{ml}^{(1)}(w,k) + I_{ml}^{(2)}(w,k)}{2} = -2\Gamma\left(m + \frac{3}{2}\right) \\
&\times \left(\frac{2}{ka}\right)^{m-(3/2)} \Re\left(i \int_0^{\pi/2} J_0(kwe^{i\vartheta}) \frac{e^{i((3/2)-m)\vartheta}}{\sqrt{e^{2i\vartheta}-1}} \right. \\
&\times \left. \{J_{m+(1/2)}(kae^{i\vartheta}) + iY_{m+(1/2)}(kae^{i\vartheta})\} d\vartheta\right). \tag{34}
\end{aligned}$$

5. Expansion of the Bessel functions and final solution of the imaginary integral

A series expansion¹⁷ of the Neumann function in Eq. (34) is given by

$$Y_{m+1/2}(kae^{i\vartheta}) = \sum_{r=0}^R \left(\frac{ka}{2}\right)^{2r-m-(1/2)} \frac{(-1)^{r+m} e^{i\vartheta(2r-m-(1/2))}}{r!\Gamma(r-m+1/2)}. \tag{35}$$

Letting $\sin \vartheta = e^{i\vartheta}$ in Eqs. (17) and (18) and substituting these together with Eq. (35) in Eq. (34) gives

$$\begin{aligned}
I_{ml}(w,k) &= -2 \sum_{q=0}^Q \sum_{r=0}^R \Re\left(\left(\frac{w}{a}\right)^{2q} \frac{(-1)^{q+r} \Gamma(m+3/2)}{(q!)^2 r! \Gamma(r+m+3/2)} \right. \\
&\times \left(\frac{ka}{2}\right)^{2(q+r+1)} i \int_0^{\pi/2} \frac{e^{2i\vartheta(q+r+1)}}{\sqrt{e^{2i\vartheta}-1}} d\vartheta \\
&- \frac{(-1)^{q+r+m} \Gamma(m+3/2)}{(q!)^2 r! \Gamma(r-m+1/2)} \left(\frac{w}{a}\right)^{2q} \\
&\times \left(\frac{ka}{2}\right)^{2(q+r-m)} \int_0^{\pi/2} \frac{e^{2i\vartheta(q+r-m+(1/2))}}{\sqrt{e^{2i\vartheta}-1}} d\vartheta \Big). \tag{36}
\end{aligned}$$

Solution of the integrals in Eq. (36) is enabled by use of the following identity:¹⁸

$$\int_0^{\pi/2} \frac{e^{2i\gamma\vartheta}}{\sqrt{e^{2i\vartheta}-1}} d\vartheta = \frac{1}{2\gamma} \left(\frac{\sqrt{\pi}\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)} - {}_2F_1\left(\frac{1}{2}, \gamma; \gamma+1; -1\right) e^{i\pi\gamma} \right). \tag{37}$$

Evaluating the integral over ϑ yields

$$\begin{aligned}
I_{ml}(w,k) &= 2\Gamma\left(m + \frac{3}{2}\right) \Re\left(\sum_{q=0}^Q \sum_{r=0}^R F_Y(q,r,m) \right. \\
&\times \left(\frac{ka}{2}\right)^{2(q+r-m)+1} \left(\frac{w}{a}\right)^{2q} - iF_J(q,r,m) \\
&\times \left.\left(\frac{ka}{2}\right)^{2(q+r+1)} \left(\frac{w}{a}\right)^{2q}\right), \tag{38}
\end{aligned}$$

where the subfunctions F_Y and F_J are given by

$$\begin{aligned}
F_Y(q,r,m) &= (-1)^{q+r+m} \left(\frac{{}_2F_1(1/2, \alpha; \alpha+1; -1) e^{i\pi\alpha}}{2\alpha(q!)^2 r! \Gamma(r-m+1/2)} \right. \\
&\quad \left. - \frac{\sqrt{\pi}\Gamma(\alpha)}{2(q!)^2 r! \Gamma(r-m+1/2)\Gamma(\alpha+1/2)} \right), \tag{39} \\
F_J(q,r,m) &= (-1)^{q+r} \left(\frac{{}_2F_1(1/2, \beta; \beta+1; -1) e^{i\pi\beta}}{2\beta(q!)^2 r! \Gamma(r+m+3/2)} \right. \\
&\quad \left. - \frac{\sqrt{\pi}\Gamma(\beta)}{2(q!)^2 r! \Gamma(r+m+3/2)\Gamma(\beta+1/2)} \right), \tag{40}
\end{aligned}$$

where

$$\alpha = q + r - m + 1/2, \tag{41}$$

$$\beta = q + r + 1. \tag{42}$$

However, for integer values of q and r , $iF_J(q,r,m)$ is purely imaginary and therefore makes no contribution to the real part of $I_{ml}(w,k)$. Similarly, the $e^{i\pi(q+r-m+1/2)}$ term of $F_Y(q,r,m)$ is also purely imaginary and can therefore be excluded. Thus, the final result can be written

$$\begin{aligned}
I_{ml}(w,k) &= -\sqrt{\pi} \sum_{q=0}^Q \sum_{r=0}^R \\
&\times \frac{(-1)^{q+r+m} \Gamma(m+3/2) \Gamma(q+r-m+1/2)}{(q!)^2 r! \Gamma(r-m+1/2) \Gamma(q+r-m+1)} \\
&\times \left(\frac{ka}{2}\right)^{2(q+r-m)+1} \left(\frac{w}{a}\right)^{2q}. \tag{43}
\end{aligned}$$

F. Calculation of the power series coefficients (final set of simultaneous equations)

Truncating the infinite power series in (11) to order M and equating the coefficients of $(w/a)^{2q}$ yields the final set of $M+1$ simultaneous equations in τ_m as follows:

$$\sum_{m=0}^M ({}_m\mathbf{P}_q(ka) + i{}_m\mathbf{T}_q(ka)) \tau_m = \delta_{q0}, \tag{44}$$

where \mathbf{P} shall be named the *Spence* function as defined by

$${}_m\mathbf{P}_q(ka) = \sqrt{\pi} \sum_{r=0}^M \frac{(-1)^{q+r} (ka/2)^{2(q+r+1)}}{(q!)^2 r! (m+3/2)_r (q+r+1)_{1/2}}, \tag{45}$$

and \mathbf{T} shall be named the *Stenzel* function as defined by

$${}_m\mathbf{T}_q(ka) = \sqrt{\pi} \sum_{r=0}^M \frac{(-1)^{q+r+m} (ka/2)^{2(q+r-m)+1}}{(q!)^2 r! (m+3/2)_{r-2m-1} (q+r-m+1/2)_{1/2}}, \tag{46}$$

δ_{q0} is the Kronecker delta function and $(x)_n$ the Pochhammer symbol.¹⁹ \mathbf{P} and \mathbf{T} are the monopole counterparts to the dipole cylindrical wave functions¹² \mathbf{B} and \mathbf{S} . These equations are then solved for $q=0, 1, 2, \dots, M-1, M$.

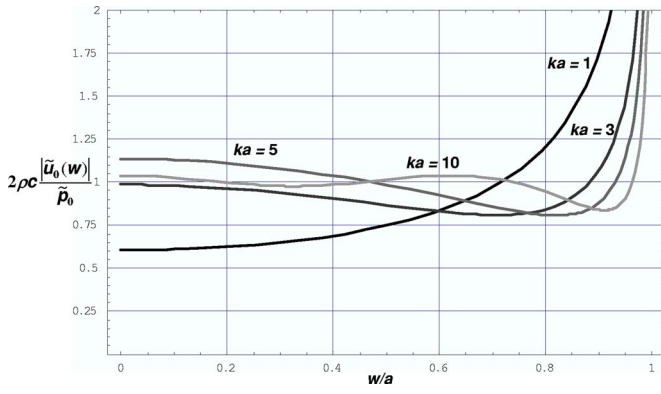


FIG. 3. Normalized surface velocity magnitude of the resilient disk.

G. Disk velocity

From Eq. (12) it follows that

$$\tilde{A}_m = \frac{\tau_m(m+1/2)\tilde{p}_0}{2\rho c}. \quad (47)$$

After substituting this in Eq. (2), the normalized disk velocity can be written as

$$\frac{2\rho c\tilde{u}_0(w_0)}{\tilde{p}_0} = \sum_{m=0}^M \tau_m \left(m + \frac{1}{2}\right) \left(1 - \frac{w_0^2}{a^2}\right)^{m-(1/2)}, \quad 0 \leq w_0 \leq a. \quad (48)$$

The magnitude and phase of the normalized velocity are shown in Figs. 3 and 4, respectively, for various values of ka .

H. Radiation admittance

The total volume velocity \tilde{U}_0 produced by the disk can be found by integrating the disk velocity from (48) over the surface of the disk as follows:

$$\tilde{U}_0 = \int_0^{2\pi} \int_0^a \tilde{u}_0(w_0)w_0 dw_0 d\phi_0 = \frac{S\tilde{p}_0}{2\rho c} \sum_{m=0}^M \tau_m, \quad (49)$$

where S is the area of the disk given by $S = \pi a^2$. The acoustic radiation admittance is then given by

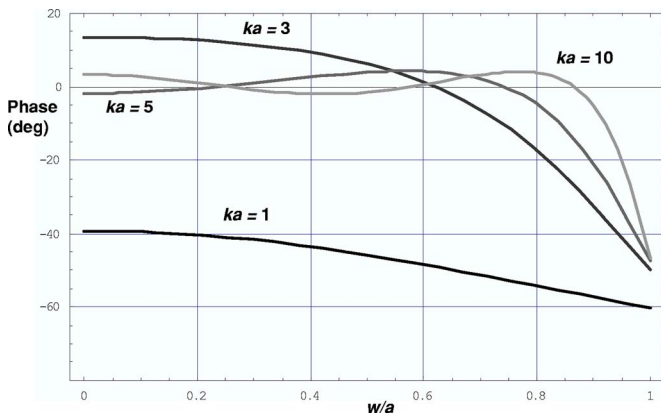


FIG. 4. Normalized surface velocity phase of the resilient disk.

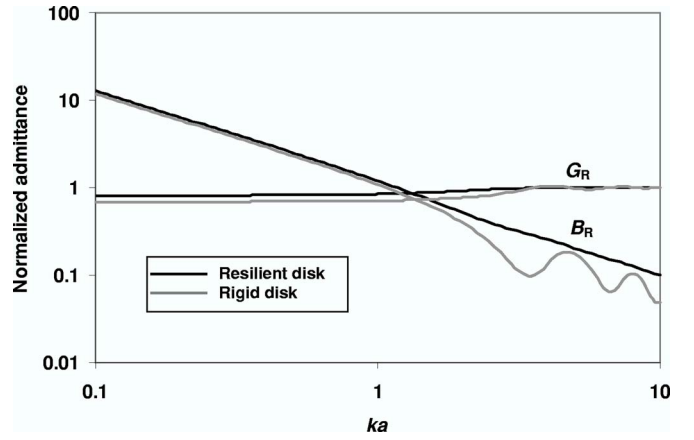


FIG. 5. Normalized radiation admittances of the rigid and resilient disk.

$$y_{ar} = \frac{\tilde{U}_0}{\tilde{p}_0} = \frac{S\tilde{u}_0}{\tilde{p}_0} = \frac{S}{2\rho c}(G_R + iB_R), \quad (50)$$

where G_R is the normalized *conductance* given by

$$G_R = \sum_{m=0}^M \Re(\tau_m) \approx \frac{8}{\pi^2}, \quad ka < 0.5, \quad (51)$$

and B_R is the normalized *susceptance* given by

$$B_R = \sum_{m=0}^M \Im(\tau_m) \approx \frac{4}{\pi ka}, \quad ka < 0.5. \quad (52)$$

The real and imaginary admittances G_R and B_R are plotted in Fig. 5 along with the actual admittance of a rigid disk for comparison.

I. Far-field pressure response

In the case of the far-field response, it is more convenient to use spherical coordinates so that the far-field polar responses can be obtained directly. Rayleigh's far-field approximation¹⁴ is ideal for this purpose,

$$g(r, \theta, \phi | w_0, \phi_0, z_0) = \frac{1}{4\pi r} e^{-ik[r-w_0 \sin \theta \cos(\phi-\phi_0)-z_0 \cos \theta]}, \quad (53)$$

which is inserted, together with Eqs. (1) and (2), in the following monopole part of the Kirchhoff-Helmholtz boundary integral formula in spherical coordinates:

$$\begin{aligned} \tilde{p}(r, \theta, \phi) &= 2 \int_0^{2\pi} \int_0^a g(r, \theta, \phi | w_0, \phi_0, z_0) |_{z_0=0+} \\ &\times \frac{\partial}{\partial z_0} \tilde{p}(w_0, \phi_0, z_0) |_{z_0=0+} w_0 dw_0 d\phi_0. \end{aligned} \quad (54)$$

After integrating over the surface of the disk [while letting $\phi = \pi/2$ so that $\cos(\phi - \phi_0) = \sin \phi_0$], the far-field pressure is given by

$$\tilde{p}(r, \theta) = -\frac{ia\tilde{p}_0}{4r} e^{-ikr} D(\theta), \quad (55)$$

where the following identities have been used:¹⁷

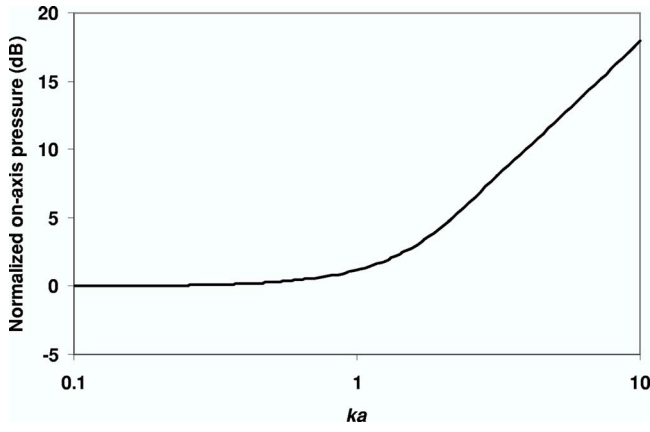


FIG. 6. Normalized far-field on-axis response of the resilient disk.

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ikw_0 \sin \theta \sin \phi_0} d\phi_0 = J_0(kw_0 \sin \theta), \quad (56)$$

together with Eq. (9), where $\mu = k \sin \theta$. The directivity function $D(\theta)$ is given by

$$D(\theta) = ka \sum_{m=0}^M \tau_m \Gamma\left(m + \frac{3}{2}\right) \left(\frac{2}{ka \sin \theta}\right)^{m+(1/2)} J_{m+(1/2)} \times (ka \sin \theta). \quad (57)$$

The on-axis pressure is obtained by setting $\theta=0$ in Eq. (53) before inserting it into Eq. (54). This results in an integral which is similar to the one for the radiation admittance in Eq. (49). Hence,

$$D(0) = ka \sum_{m=0}^M \tau_m \approx \begin{cases} 4i/\pi, & ka < 0.5 \\ ka, & ka > 2. \end{cases} \quad (58)$$

It is worth noting that the on-axis response is related to the radiation admittance by $D(0) = ka(G_R + iB_R)$. The asymptotic expression for low-frequency on-axis pressure is then simply

$$\tilde{p}(r, 0) \approx \frac{a}{\pi r} \tilde{p}_0 e^{-ikr}, \quad ka < 0.5, \quad (59)$$

and likewise at high frequencies

$$\tilde{p}(r, 0) \approx i \frac{ka^2}{4r} \tilde{p}_0 e^{-ikr}, \quad ka > 2, \quad (60)$$

which is the same as for a resilient disk in free space at all frequencies. The on-axis response is shown in Fig. 6, calculated from the magnitude of $D(0)$. The normalized directivity function $20 \log_{10}(|D(\theta)|/|D(0)|)$ is plotted in Fig. 7 for various values of ka .

J. Near-field pressure when the distance from the center of the disk to the observation point is greater than the diaphragm's radius

Using the monopole part of the Kirchhoff-Helmholtz boundary integral formula,¹⁴ the sound pressure at the observation point P can be written as

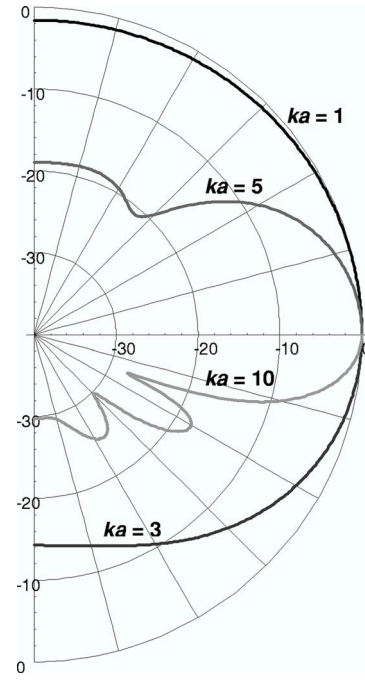


FIG. 7. Normalized far-field directivity function of the resilient disk.

$$\tilde{p}(r, \theta) = 2 \int_0^{2\pi} \int_0^a g(r, \theta|w_0, \phi_0) \times \frac{\partial}{\partial z_0} \tilde{p}(w_0, z_0) \Big|_{z_0=0} w_0 dw_0 d\phi_0, \quad (61)$$

where a is the radius of the disk and $g(r, \theta|w_0, \phi)$ is the Green's function. The Green's function is defined by

$$g(r, \theta|w_0, \phi_0) = \frac{e^{-ikr_0}}{4\pi r_0}, \quad (62)$$

where

$$r_0 = \sqrt{r^2 + w_0^2 - 2rw_0 \cos \phi_0 \sin \theta}, \quad (63)$$

so that Eq. (61) becomes the Rayleigh integral. The Green's function of Eq. (62) can be expanded using the following formula, which is a special case of Gegenbauer's addition theorem:¹⁷

$$g(r, \theta|w_0, \phi_0) = -\frac{ik}{4\pi} \sum_{p=0}^{\infty} (2p+1) h_p^{(2)} \times (kr) j_p(kw_0) P_p(\cos \phi_0 \sin \theta), \quad (64)$$

where $h_p^{(2)}$ is the spherical Hankel function.^{19,14} Substituting Eqs. (64), (1), (2), and (47) in Eq. (61) enables the integrals in Eq. (61) to be separated as follows:

$$\begin{aligned} p(r, \theta) &= -\frac{k^2 \tilde{p}_0}{2\pi} \sum_{p=0}^{\infty} \left(p + \frac{1}{2}\right) h_p^{(2)}(kr) \sum_{m=0}^{\infty} \tau_m (m+1/2) \\ &\times \int_0^a \left(1 - \frac{w_0^2}{a^2}\right)^{m-(1/2)} j_p(kw_0) w_0 dw_0 \\ &\times \int_0^{2\pi} P_p(\cos \phi_0 \sin \theta) d\phi_0. \end{aligned} \quad (65)$$

The Legendre function P_p can be expanded using the following addition theorem¹⁷ (after setting one of the three angles in the original formula to $\pi/2$):

$$P_p(\cos \phi_0 \sin \theta) = P_p(0)P_p(\cos \theta) + 2 \sum_{q=1}^{\infty} (-1)^q P_p^{-q}(0) \times P_p^q(\cos \theta) \cos q \phi_0, \quad (66)$$

which leads to the identity

$$\int_0^{2\pi} P_p(\cos \phi_0 \sin \theta) d\phi_0 = 2\pi P_p(0)P_p(\cos \theta) + 2 \sum_{q=1}^{\infty} (-1)^q P_p^{-q}(0) P_p^q(\cos \theta) \times \frac{\sin 2\pi q}{q} = 2\pi P_p(0)P_p(\cos \theta), \quad \text{for integral } q. \quad (67)$$

It is also noted that¹⁷

$$P_{2p}(0) = \frac{\sqrt{\pi}}{p! \Gamma((1/2) - p)} = \frac{(-1)^p \Gamma(p + (1/2))}{\sqrt{\pi} p!}, \quad (68)$$

and

$$P_{2p+1}(0) = 0, \quad (69)$$

so that, after substituting Eqs. (67)–(69) in Eq. (65) and excluding the odd terms, only the radial integral remains as follows:

$$p(r, \theta) = -\sqrt{\pi} k^2 \tilde{p}_0 \sum_{p=0}^{\infty} \frac{(2p + (1/2)) h_{2p}^{(2)}(kr)}{p! \Gamma((1/2) - p)} \times \sum_{m=0}^{\infty} \tau_m(m + 1/2) \times \int_0^a \left(1 - \frac{w_0^2}{a^2}\right)^{m-(1/2)} \times j_{2p}(kw_0) w_0 dw_0 P_{2p}(\cos \theta). \quad (70)$$

With the help of the following identity:¹⁸

$$\int_0^a \left(1 - \frac{w_0^2}{a^2}\right)^{m-(1/2)} j_{2p}(kw_0) w_0 dw_0 = \frac{\sqrt{\pi}}{k^2(m + (1/2))_{p+1}} \times \frac{p!}{\Gamma(2p + (3/2))} \left(\frac{ka}{2}\right)^{2p+2} \times {}_1F_2\left(p + 1; p + m + \frac{3}{2}, 2p + \frac{3}{2}; -\frac{k^2 a^2}{4}\right), \quad (71)$$

the solution is given by

$$p(r, \theta) = -\tilde{p}_0 \sum_{m=0}^M \tau_m \sum_{p=0}^P \frac{(-1)^p \Gamma(p + (1/2)) h_{2p}^{(2)}(kr) P_{2p}(\cos \theta)}{\Gamma(2p + (1/2)) (m + (3/2))_p} \times \left(\frac{ka}{2}\right)^{2p+2} {}_1F_2\left(p + 1; p + m + \frac{3}{2}, 2p + \frac{3}{2}; -\frac{k^2 a^2}{4}\right). \quad (72)$$

This expansion converges providing $r \geq a$. Let an error function be defined by

$$\varepsilon_P(r, \theta) = \frac{|p_{2P}(r, \theta) - p_P(r, \theta)|}{|p_P(r, \theta)|}, \quad (73)$$

so that the pressure obtained with an expansion limit $2P$ is used as a reference. The calculations were performed using 30-digit precision with $P \approx 2ka$ and $M = P$. This produced values of ε typically on the order of 10^{-8} except at $r = a$ where it was on the order of 0.01. At $r = a$ and $\theta = \pi/2$ (i.e., the disk rim), there is a singularity which was avoided by the use of a small offset. For $ka < 1$, values $M = P = 2$ were used.

K. Near-field pressure when the distance from the center of the disk to the observation point is less than the diaphragm's radius

1. The near-field pressure as an integral expression

The simplest way to derive an expression for the immediate near-field pressure is to use the King integral. Applying the expression for \tilde{A}_m in Eq. (47) to Eq. (8), the near-field pressure can be written

$$\tilde{p}(w, z) = \tilde{p}_0 \sum_{m=0}^{\infty} \tau_m \Gamma\left(m + \frac{3}{2}\right) (I_{\text{Fin}}(m, w, z) - i I_{\text{Inf}}(m, w, z)), \quad (74)$$

where

$$I_{\text{Fin}}(m, w, z) = \frac{ka}{2} \int_0^k \left(\frac{2}{a\mu}\right)^{m-(1/2)} J_{m+(1/2)}(a\mu) \times J_0(w\mu) \frac{e^{-iz\sqrt{k^2-\mu^2}}}{\sqrt{k^2-\mu^2}} d\mu \quad (75)$$

and

$$I_{\text{Inf}}(m, w, z) = \frac{ka}{2} \int_k^{\infty} \left(\frac{2}{a\mu}\right)^{m-(1/2)} J_{m+(1/2)}(a\mu) \times J_0(w\mu) \frac{e^{-z\sqrt{\mu^2-k^2}}}{\sqrt{\mu^2-k^2}} d\mu. \quad (76)$$

2. Solution of the finite integral

Substituting $\mu = k\sqrt{1-t^2}$ in Eq. (75) in order to simplify the exponent yields

$$I_{\text{Fin}}(m, w, z) = \left(\frac{2}{ka}\right)^{m-(3/2)} \times \int_0^1 \frac{J_{m+(1/2)}(ka\sqrt{1-t^2}) J_0(kw\sqrt{1-t^2})}{(1-t^2)^{(m/2)+(1/4)}} \times e^{-ikzt} d\mu. \quad (77)$$

The Bessel functions in Eq. (77) can then be expanded using the following Lommel expansion:²⁰

$$\frac{J_n(ka\sqrt{1-t^2})}{(1-t^2)^{n/2}} = \sum_{m=0}^{\infty} \left(\frac{ka}{2}\right)^m t^{2m} \frac{J_{n+m}(ka)}{m!}, \quad (78)$$

which leads to

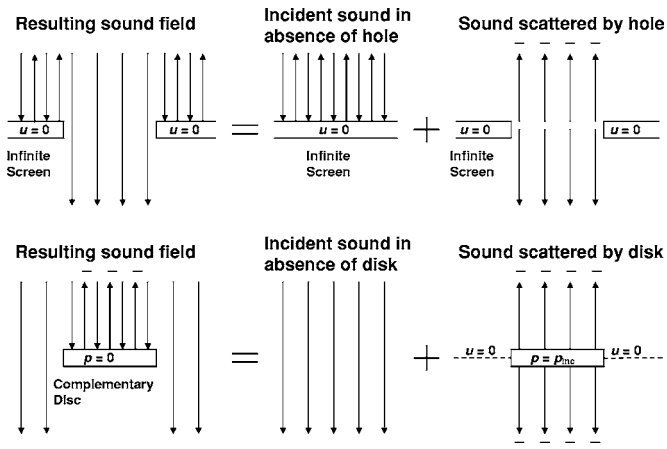


FIG. 8. Babinet's principle.

$$I_{\text{Fin}}(m, w, z) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left(\frac{ka}{2} \right)^{p-m+(3/2)} \times \left(\frac{kw}{2} \right)^q \frac{J_{p+m+(1/2)}(ka) J_q(kw)}{p!q!} \times \int_0^1 e^{-ikzt} t^{2(p+q)} d\mu. \quad (79)$$

The integral in Eq. (79) can be solved using the identity¹⁷

$$\int_0^1 e^{-ikzt} t^{2(p+q)} dt = \frac{\gamma(2p+2q+1, ikz)}{(ikz)^{2(p+q)+1}}, \quad (80)$$

where γ is the incomplete gamma function. Inserting Eq. (80) in Eq. (79) and truncating the summation limits gives the final solution to Eq. (75) as follows:

$$I_{\text{Fin}}(m, w, z) = - \sum_{p=0}^P \sum_{q=0}^Q \frac{1}{p!q!(ikz)^{2(p+q)+1}} \left(\frac{ka}{2} \right)^{p-m+(3/2)} \times \left(\frac{kw}{2} \right)^q \times J_{p+m+(1/2)}(ka) J_q(kw) \gamma(2p+2q+1, ikz). \quad (81)$$

This solution converges for all $w > 0$ and $z > 0$, and was used for the region $0 < w^2 + z^2 < a^2$, with a small offset at $z=0$. On the axis of symmetry ($w=0$), only the zeroth term of the expansion in q remains and the solution reduces to a single expansion,

$$I_{\text{Fin}}(m, 0, z) = - \sum_{p=0}^P \frac{1}{p!(ikz)^{2p+1}} \left(\frac{ka}{2} \right)^{p-m+(3/2)} J_{p+m+(1/2)} \times (ka) \gamma(2p+1, ikz), \quad (82)$$

which converges for $z > 0$.

3. Solution of the infinite integral

For the finite integral, it was sufficient to expand both Bessel functions with the Lommel expansion. In the case of

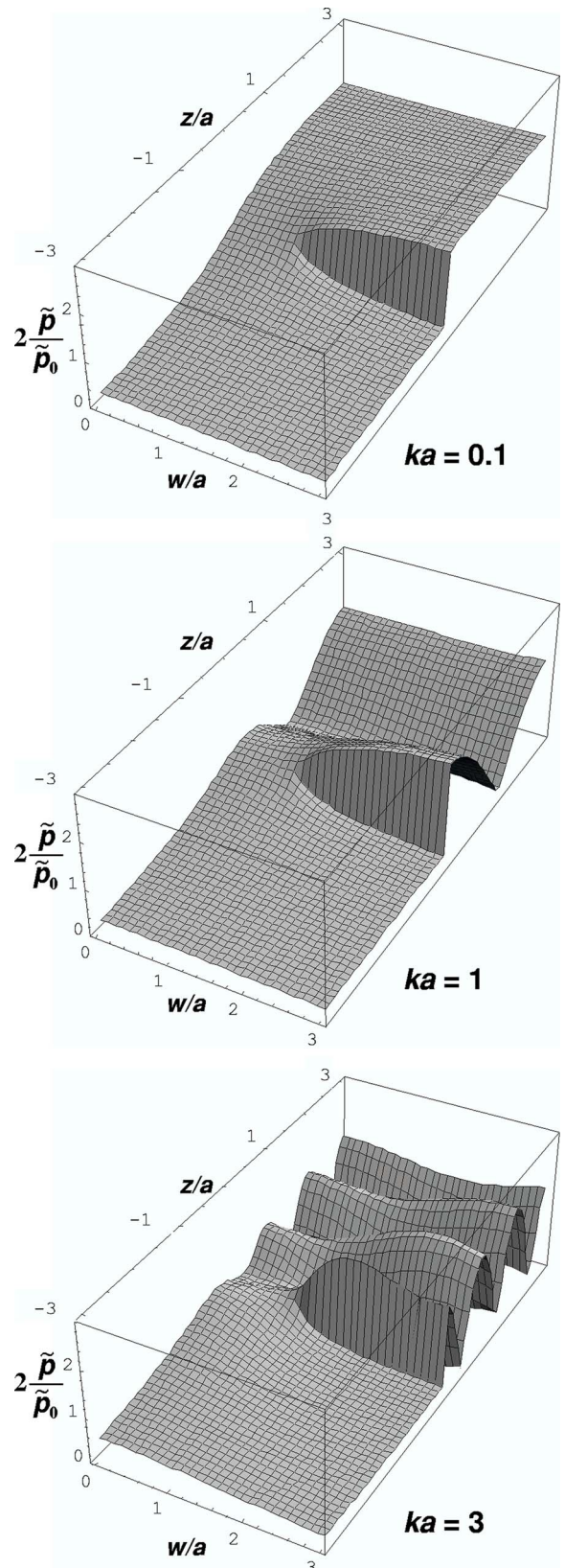


FIG. 9. Near-field pressure of a plane wave passing through a hole in an infinite screen for $ka=0.1, 1$, and 3 .

the infinite integral, this would lead to a solution which only converges for $z \geq w+a$. Therefore, in order to ensure convergence, one of the Bessel functions is to be expanded using Gegenbauer's summation theorem¹⁷ as follows:

$$\frac{J_n(kb\sqrt{t^2+1})}{(t^2+1)^{n/2}} = \Gamma(n) \left(\frac{2}{kbt}\right)^n \sum_{m=0}^{\infty} (m+n) J_{m+n}(kbt) \times J_{m+n}(kb) C_m^n(0), \quad (83)$$

where C_m^n is the Gegenbauer¹⁷ polynomial given by

$$C_m^n(0) = \frac{\cos(m\pi/2)\Gamma(n+m/2)}{(m/2)!\Gamma(n)}, \quad (84)$$

where n is a positive real nonzero integer. Inserting Eq. (84) in Eq. (83) and noting that, due to the cosine term, all odd terms of the Gegenbauer polynomial are zero yields

$$\frac{J_n(ka\sqrt{t^2+1})}{(t^2+1)^{n/2}} = \left(\frac{2}{kat}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (2m+n)\Gamma(m+n) \times J_{2m+n}(ka) J_{2m+n}(kat). \quad (85)$$

Substituting $\mu = k\sqrt{t^2+1}$ in Eq. (76) in order to simplify the exponent yields

$$I_{\text{Inf}}(m, w, z) = \left(\frac{2}{ka}\right)^{m-(3/2)} \int_0^{\infty} \frac{J_{m+(1/2)}(ka\sqrt{t^2+1})}{(t^2+1)^{(m/2)+(1/4)}} \times J_0(kw\sqrt{t^2+1}) e^{-kzt} dt. \quad (86)$$

Expanding $J_0(kw\sqrt{t^2+1})$ in Eq. (86) with Eq. (78) and $J_{m+(1/2)}[ka\sqrt{t^2+1}]$ with Eq. (85) yields

$$I_{\text{Inf}}(m, w, z) = \left(\frac{2}{ka}\right)^{2m-1} \times \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left(\frac{kw}{2}\right)^q \frac{(-1)^{p+q} (2p+m+(1/2))}{p!q!} \times \Gamma(p+m+(1/2)) J_{2q+m+(1/2)}(ka) J_q(kw) \times \int_0^{\infty} J_{2q+m+(1/2)}(kat) e^{-kzt} t^{2q-m-(1/2)} dt. \quad (87)$$

The integral in Eq. (87) can be solved using the following identity:¹⁷

$$\int_0^{\infty} e^{-kzt} J_{2p+m+(1/2)}(kat) t^{2q-m-(1/2)} dt = \frac{\Gamma(2p+2q+1)}{(k^2z^2+k^2a^2)^{q-(m/2)+(1/4)}} P_{2q-m-(1/2)}^{-2p-m-(1/2)} \left(\frac{z}{\sqrt{z^2+a^2}}\right), \quad (88)$$

so that, after truncating the summation limits, the final solution is given by

$$I_{\text{Inf}}(m, w, z) = \left(\frac{2}{ka}\right)^{2m-1} \sum_{p=0}^P \sum_{q=0}^Q \frac{(-1)^{p+q} (2p+m+1/2)}{q!(k^2z^2+k^2a^2)^{q-(m/2)+(1/4)}} \times (p+1)_{m-(1/2)} \Gamma(2p+2q+1) \left(\frac{kw}{2}\right)^q \times J_{2p+m+(1/2)}(ka) J_q(kw) P_{2q-m-(1/2)}^{-2p-m-(1/2)} \times \left(\frac{z}{\sqrt{z^2+a^2}}\right). \quad (89)$$

This ‘‘paraxial’’ expression converges for $w^2 < a^2 + z^2$ and

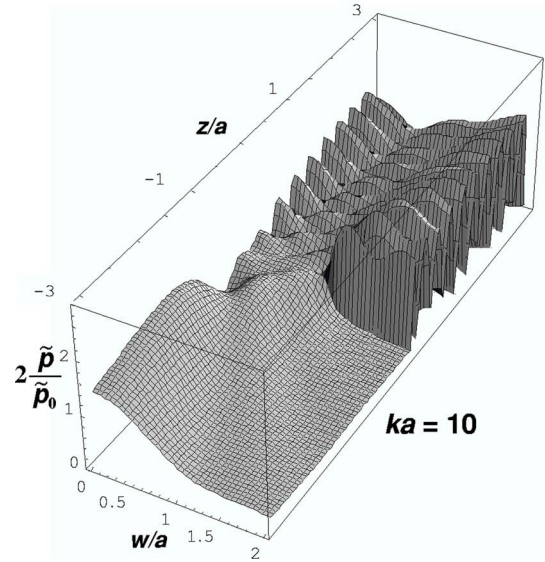
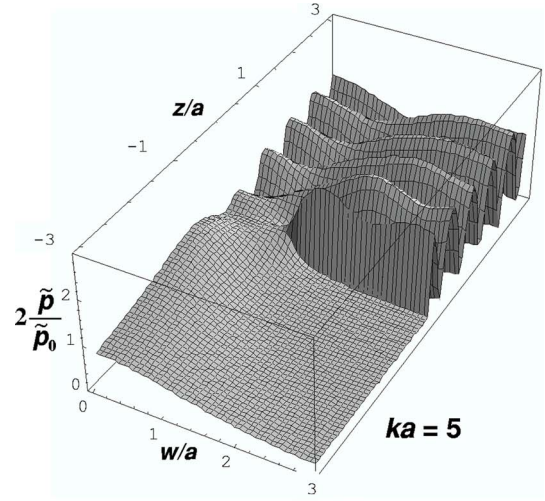


FIG. 10. Near-field pressure of a plane wave passing through a hole in an infinite screen for $ka=5$ and 10.

was used for the region $0 < w^2 + z^2 < a^2$, with a small offset at $z=0$. At $w \rightarrow a$ and $z \rightarrow 0$ (i.e., the disk rim), there is a singularity. On the axis of symmetry ($w=0$), only the zeroth term of the expansion in q remains and the solution reduces to a single expansion,

$$I_{\text{Inf}}(m, 0, z) = \left(\frac{2}{ka}\right)^{2m-1} \sum_{p=0}^P \frac{(-1)^p (2p+m+1/2)}{(k^2z^2+k^2a^2)^{-(m/2)+(1/4)}} \times (p+1)_{m-(1/2)} \Gamma(2p+1) \times J_{2p+m+(1/2)} \times (ka) P_{-m-(1/2)}^{-2p-m-(1/2)} \left(\frac{z}{\sqrt{z^2+a^2}}\right), \quad (90)$$

which converges for $z > 0$. The calculations for Eqs. (74), (81), and (89) were performed using 30-digit precision with $P \approx 2ka$, $Q \approx 4kw$, and $M = P$. This produced values of ε typically of the order of 10^{-8} . For $ka < 1$, values $M = P = 2$ and $Q = 4$ were used.

III. BABINET'S PRINCIPLE

Babinet's principle,²¹ as developed by Bouwkamp,²² states that the diffraction pattern resulting from the transmission of a plane wave through a hole in an infinite rigid screen (i.e., with infinite surface impedance) is equivalent to that produced by the scattering of the same incident wave by the complementary resilient disk. Furthermore, the scattered wave is identical to that produced if the disk itself were radiating in an infinite baffle, providing the surface pressure of the disk is equal to the pressure of the incident wave at the surface of the disk or hole in the absence of any obstacle. This is illustrated in Fig. 8. For clarity, the diagram portrays the scattering of a sound wave at some very high frequency where there is minimal diffraction. However, the principle applies at all frequencies. The resulting sound field is given by

$$\tilde{p}(\mathbf{r}) = \tilde{p}_{\text{Inc}}(\mathbf{r}) + \tilde{p}_{\text{Scat}}(\mathbf{r}), \quad (91)$$

where $\tilde{p}_{\text{Inc}}(\mathbf{r})$ is the incident sound field in the absence of a hole or disk given in terms of the velocity potential $\tilde{\Psi}$ by

$$\tilde{p}_{\text{Inc}}(z) = \begin{cases} -ik\rho c\tilde{\Psi}(e^{ikz} + e^{-ikz}), & \text{bright side of screen} \\ 0, & \text{dark side of screen} \\ -ik\rho c\tilde{\Psi}e^{ikz}, & \text{without disk (or screen)} \end{cases} \quad (92)$$

using Eq. (74) for $\tilde{p}_{\text{Scat}}(w, z)$ (immediate near field), or Eq. (72) for $\tilde{p}_{\text{Scat}}(r, \theta)$ (intermediate near field), or Eqs. (55) and (57) for $\tilde{p}_{\text{Scat}}(r, \theta)$ (far field). Also, it can be stated that the volume velocity \tilde{U} at the disk due to an incident wave is given by

$$\tilde{U} = y_{\text{ar}}\tilde{p}_{\text{Inc}}. \quad (93)$$

The radiation admittance y_{ar} is given by Eqs. (50)–(52). The results are shown in Figs. 9–11 for various values of ka . The pressure is plotted against the axisymmetric cylindrical ordinates w and z using

$$r = \sqrt{w^2 + z^2}, \quad (94)$$

$$\theta = \arctan w/z, \quad (95)$$

and the parameter P_{norm} is given by

$$P_{\text{norm}} = \left| \frac{\tilde{p}(w, z)}{\rho c \tilde{u}_0} \right|. \quad (96)$$

IV. CONCLUSIONS

A set of solutions has been obtained for axisymmetric sources in infinite baffles which appear to be relatively compact and can be calculated fairly easily without numerical problems. As an example, the radiation characteristics of a resilient disk have been calculated. By applying Babinet's principle, this solution has also been used to calculate the pressure field of a plane wave passing through a circular hole in an infinite screen.

The radiation conductance (a.k.a. transmission coefficient), velocity magnitude and phase, and far-field directivity

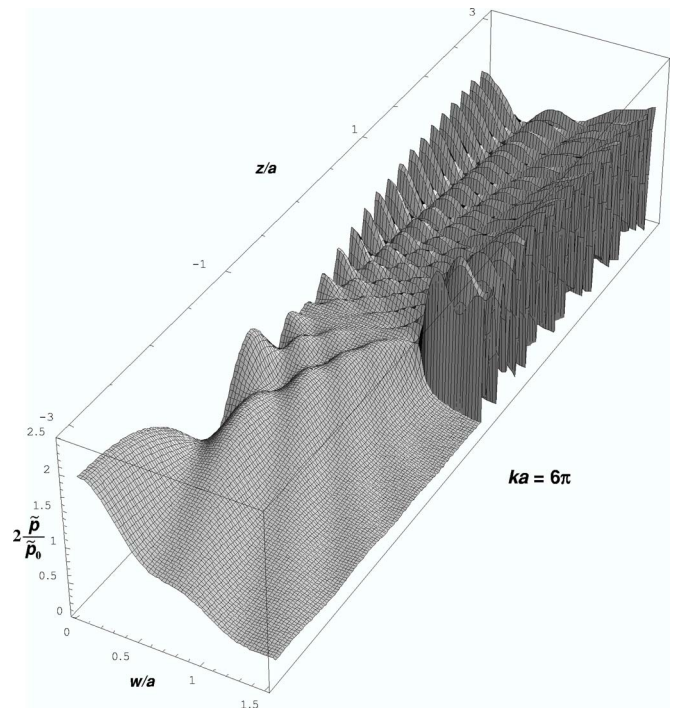


FIG. 11. Near-field pressure of a plane wave passing through a hole in an infinite screen for $ka=6\pi$.

function of the resilient disk in Figs. 5, 3, 4, and 7 respectively, show good agreement with the calculations of Spence^{23,24} and the formulas provided here are intended to provide a simple alternative calculation method. For instance, Eqs. (44)–(46) and (50)–(52) for the radiation admittance appear to be relatively compact, thus eliminating the need for complicated spheroidal wave functions.

Although these derivations can be applied to membranes, plates, or shallow shells, to do so here would result in an overly long text. However, it would be interesting to see the cylindrical wave functions and pressure field expansions applied to fluid-structure coupled problems such as loudspeaker diaphragms, for example.

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