Expansions for the radiation impedance of a rectangular piston in an infinite baffle

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Relatively compact analytical expressions in the form of fast-converging expansions are derived for the radiation resistance and reactance of a rectangular rigid piston in an infinite rigid baffle, which are computationally efficient at high frequencies or large aspect ratios and yield simple approximations (asymptotic expressions) at low frequencies. Plots of the normalized radiation resistance and reactance are shown for various aspect ratios with constant width as well as constant area. Comparisons are also made with the impedance of an elliptic piston.
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I. INTRODUCTION

The radiation impedance of a rectangular rigid piston in an infinite rigid baffle is one of the fundamental problems of sound radiation, along with that of a circular piston. It is applicable to rectangular loudspeaker diaphragms, horns, and sound holes. Not surprisingly it has received attention from a number of authors over the years using a variety of different approaches. Stenzel[1] and Bank and Wright[2] use the Rayleigh integral with the traditional Euclidean Green's function in a moveable-origin rectangular coordinate system. Surprisingly, whereas Stenzel evaluates the integrals analytically, Bank and Wright calculate the quadruple integral numerically without any attempt to simplify it. Chetaev[3] reduces the quadruple integral to a single one which Burnett and Soroka[4] calculate numerically to produce tables accurate to seven decimal places using a specially developed technique[5] for highly oscillatory integrands.

Arase[6] uses numerical integration to calculate the mutual radiation impedance of two square pistons in an infinite baffle, while Lee and Seo[7] calculate the radiation impedance of a single square piston. Levine[8] reduces the quadruple integral for the radiation efficiency of a rectangular panel with a non-uniform modal velocity distribution down to a single one.

Morse and Ingard[9] also use the Rayleigh integral but with a Fourier Green's function in rectangular coordinates, which is the approach used in this paper. Confusingly, they do not reduce the Green's function from a triple infinite-integral form to the simpler double one (as presented here) until *after* inserting it into the Rayleigh integral. Then they do not evaluate (solve) the integrals but simply state that the sinc terms can be expanded and integrated term by term. Although valid for the radiation resistance, this statement is somewhat misleading in the case of the radiation reactance because the infinite integral leads to diverging results when evaluated this way. The infinite integral should be evaluated analytically before expanding the integrand in order to evaluate the remaining finite integral, as demonstrated in this paper. Not surprisingly, the form of the integral equation obtained when using this approach is equivalent to that derived using Bouwkamp's theorem[10] whereby the impedance is obtained from the directivity pattern by integrating the pressure over a hemisphere in the far field, as previously demonstrated in the case of a rectangular piston.[11] In fact the universality of Bouwkamp's theorem is demonstrated by Mechel's application of it to an elliptic radiator,[12] as reproduced here for comparison.

An alternative approach is used by Stepanishen,[13] based on the elegant theory of Lindemann,[14] in which the impedance is expressed as a one-dimensional Fourier transformation of its time-domain impulse response. The difficulty of this approach arises from the oscillatory integral in Eq. (22) of Ref. 13 which is only evaluated analytically for the first two terms of each of the expansions for the resistance and reactance. Further analytical solutions could be obtained through symbolic computation, but this is inconvenient. Alternately, a general solution could be obtained but this leads to complicated expressions for the resistance and reactance with eight hypergeometric functions in each. However, the low frequency approximations provided are particularly useful.

Only Stenzel[1] provides complete analytical evaluations of all the integrals involved because, in the age of mechanical computation, numerical integration of oscillatory integrands was not an option. All of the subsequent authors, most of whom appear to be unaware of his work, rely on numerical integration and so Stenzel's expressions have, until now, been computationally the most efficient.

If Stenzel's approach, using the Euclidean Green's function, could be considered analogous to Rayleigh's derivation[15] for the radiation impedance of a circular piston in an infinite baffle, then the approach in this paper, using the Fourier Green's function, could be considered analogous to

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King's derivation.[16] In other words, the idea is to demonstrate the utility of the elegant Fourier Green's function, which has found widespread use in the field of near-field acoustic holography[17] because it can be used to predict the sound field from that captured within a plane back to the source as well as forward from the plane, as demonstrated by Stepanishen and Benjamin[18] using a square vibrator with a non-uniform velocity. This approach has also been applied to a rectangular interface within a rigid baffle.[19]

The main purpose of this paper is to present expressions for the radiation resistance and reactance that are more compact and convenient to use without the need for numerical integration which requires special techniques and is therefore more complicated to program. These expressions have been found to agree numerically with those of Stenzel and the data presented by Bank and Wright[2] and Burnett and Soroka.[4] However, by using hypergeometric functions instead of Stenzel's logarithmic ones in the expression for the reactance, numerical stability is improved at high frequencies or large aspect ratios because this avoids taking the differences between very large terms. Furthermore, this derivation yields simple low-frequency asymptotic expressions which agree with those of Stepanishen.[13]

II. DERIVATION OF THE NEAR-FIELD PRESSURE AND RADIATION FORCE

A. Boundary conditions

The rectangular piston of area $S = 4a_x a_y$ shown in Fig. 1 is mounted in an infinite rigid baffle in the xy plane with its center at the origin and oscillates in the z direction with a uniform surface velocity distribution $\tilde{u}_0$, thus radiating from both sides into a homogeneous loss-free medium, where the tilde denotes a harmonically time-varying quantity or missing factor of $e^{i\omega t}$. Although the actual pressure field on one side of the xy plane is anti-symmetrical to that on the other, for our analysis it is convenient to remove the baffle altogether and model the system using a monopole "breathing" piston in free space. In other words, it is equivalent to two back-to-back pistons (with a vanishingly small distance between them) moving in opposite directions. Hence

$$\tilde{p}(x, y, z) = \tilde{p}(x, y, -z). \tag{1}$$

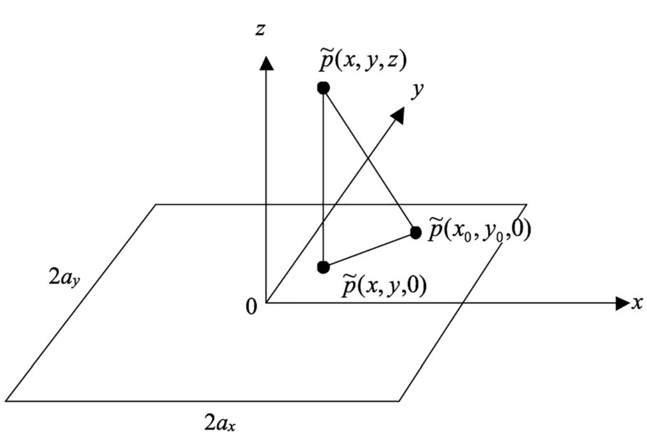

FIG. 1. Geometry of the rectangular piston.

Consequently, there is a Neumann boundary condition in the plane of the piston where these equal fields meet

$$\frac{\partial}{\partial z}\tilde{p}(x, y, z)|_{z=0} = 0, \quad \begin{cases} -\infty \le x < -a_x, & a_x < x \le \infty \\ -\infty \le y < -a_y, & a_y < y \le \infty, \end{cases} \tag{2}$$

which is satisfied automatically and thus replicates the baffle. Also, on the surface of the piston, there is the coupling condition

$$\frac{\partial}{\partial z}\tilde{p}(x, y, z)|_{z=0} = -ik\rho c\tilde{u}_0, \quad \begin{cases} -a_x \le x \le a_x \\ -a_y \le y \le a_y, \end{cases} \tag{3}$$

where k is the wave number given by $k = \omega/c = 2\pi/\lambda$, ω is the angular frequency of excitation, ρ is the density of the surrounding medium, c is the speed of sound in that medium, and λ is the wavelength.

B. Boundary integrals

The near-field pressure is given by the monopole Rayleigh integral or monopole part of the Kirchhoff-Helmholtz boundary integral taking into account the double-strength source[11]

$$\tilde{p}(x, y, z) = -2\int_{-a_y}^{a_y}\int_{-a_x}^{a_x}\frac{\partial}{\partial z}\tilde{p}(x, y, z)|_{z=0} \times g(x, y, z|x_0, y_0, z_0)|_{z_0=0}dx_0dy_0. \tag{4}$$

The Euclidean Green's function in three-dimensional rectangular coordinates is given by

$$g(x, y, z|x_0, y_0, z_0) = \frac{e^{-ikR}}{R}, \tag{5}$$

where

$$R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}. \tag{6}$$

However, we shall use the more powerful Fourier Green's function which is given by

$$g(x, y, z|x_0, y_0, z_0) = -\frac{i}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i(k_x(x-x_0)+k_y(y-y_0)+k_z(z-z_0))}}{k_z}dk_xdk_y, \tag{7}$$

where

$$k_z = \begin{cases} \sqrt{k^2 - k_x^2 - k_y^2}, & k_x^2 + k_y^2 \le k^2 \\ -i\sqrt{k_x^2 + k_y^2 - k^2}, & k_x^2 + k_y^2 > k^2. \end{cases} \tag{8}$$

A formal derivation of Eq. (7) is given in Ref. 11 and it also appears as Eq. (11) in Ref. 19. When using the Fourier Green's function, Eq. (4) represents a Fourier transform of the surface velocity distribution into k-space and the Green's function of Eq. (7) represents the inverse transform as well as the propagation of the spatial spectra in the z direction, as detailed by Eqs. (7) and (8) of Ref. 18. In this paper, the

Fourier Green's function has the advantage over the Euclidean one of losing the square root signs so that the variables become separable when evaluating the integrals.

C. Near-field pressure and radiation force

Inserting Eqs. (3) and (7) into Eq. (4) and integrating over x_0 and y_0 gives

$$\tilde{p}(x,y,z) = \frac{k\rho c\tilde{u}_0}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i(k_x x+k_y y+k_z z)}}{k_z} \times \int_{-a_x}^{a_x} e^{ik_x x_0}dx_0 \int_{-a_y}^{a_y} e^{ik_y y_0}dy_0 dk_x dk_y$$
$$= \frac{k\rho c a_x a_y \tilde{u}_0}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i(k_x x+k_y y+k_z z)}}{k_z} \times \mathrm{sinc}(k_x a_x)\mathrm{sinc}(k_y a_y)dk_x dk_y, \tag{9}$$

where we have used sinc $x = (\sin x)/x$. The total radiation force is then given by

$$\tilde{F} = \int_{-a_x}^{a_x}\int_{-a_y}^{a_y}\tilde{p}(x,y,z)|_{z=0}dxdy$$
$$= \frac{4k\rho c a_x^2 a_y^2 \tilde{u}_0}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{sinc}^2(k_x a_x)\mathrm{sinc}^2(k_y a_y)\frac{1}{k_z}dk_x dk_y. \tag{10}$$

III. RADIATION IMPEDANCE

The specific radiation impedance is the ratio of the total radiation force to the total volume velocity

$$Z_s = \frac{\tilde{F}}{\tilde{U}_0} = \frac{\tilde{F}}{4a_x a_y \tilde{u}_0}$$
$$= \frac{k\rho c a_x a_y}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{sinc}^2(k_x a_x)\mathrm{sinc}^2(k_y a_y)\frac{1}{k_z}dk_x dk_y. \tag{11}$$

By using polar coordinates, where $k_x = kt\cos\varphi$, $k_y = kt\sin\varphi$, and $dk_x dk_y = k^2 t dt d\varphi$, we reduce the double infinite integral to a single infinite one and a single finite one. Also, the infinite integral can be split into the real resistance R_s and imaginary reactance X_s so that

$$Z_s = R_s - iX_s, \tag{12}$$

where

$$R_s = \frac{4ka_x ka_y \rho c}{\pi^2}\int_0^{\pi/2}\int_0^1 \mathrm{sinc}^2(ka_x t\cos\varphi) \times \mathrm{sinc}^2(ka_y t\sin\varphi)\frac{tdtd\varphi}{\sqrt{1-t^2}}, \tag{13}$$

$$X_s = \frac{4ka_x ka_y \rho c}{\pi^2}\int_0^{\pi/2}\int_1^{\infty} \mathrm{sinc}^2(ka_x t\cos\varphi) \times \mathrm{sinc}^2(ka_y t\sin\varphi)\frac{tdtd\varphi}{\sqrt{t^2-1}}. \tag{14}$$

The solutions of these integrals are detailed in the Appendix in order to yield the following expansions for the radiation resistance and reactance:

$$R_s = \frac{\rho c}{\sqrt{\pi}}\sum_{m=0}^{M}\sum_{n=0}^{N}\frac{(-1)^{m+n}q^{2n+1}(ka_x)^{2m+2n+2}}{(2m+1)(2n+1)(m+1)!(n+1)!\Gamma\left(m+n+\frac{3}{2}\right)}, \tag{15}$$

$$X_s = \frac{\rho c}{\pi}\left(\frac{1-\mathrm{sinc}(2ka_x)+q(1-\mathrm{sinc}(2qka_x))}{qka_x} + 2\sum_{m=0}^{M}\frac{(-1)^m(ka_x)^{2m+1}}{(2m+1)m!(m+1)!}f_m(q)\right), \tag{16}$$

where

$$f_m(q) = \frac{{}_2F_1\left(1, m+\frac{1}{2}; m+\frac{3}{2}, \frac{1}{1+q^2}\right) + {}_2F_1\left(1, m+\frac{1}{2}; m+\frac{3}{2}, \frac{1}{1+q^{-2}}\right)}{(2m+1)(1+q^{-2})^{m+1/2}} + \frac{1}{2m+3}\sum_{n=0}^{m} g_{m,n}(q), \tag{17}$$

$$g_{m,n}(q) = \binom{2m+3}{2n}\sum_{p=n}^{m}\frac{(-1)^{p-n}q^{2n-1}}{(2p-1)(1+q^2)^{p-1/2}}\binom{m-n}{p-n} + \binom{2m+3}{2n+3}\sum_{p=m-n}^{m}\frac{(-1)^{p-m+n}q^{2n+2}}{(2p-1)(1+q^{-2})^{p-1/2}}\binom{n}{p-m+n}, \tag{18}$$

$q = a_y/a_x$ is the aspect ratio and ${}_2F_1$ is the hypergeometric function.[20] Although Eqs. (16)–(18) for the reactance are in the form of a triple expansion in m, n, and p, only the expansion in m is frequency-dependent. The inner terms $f_m(q)$ can be reused for each frequency step during the calculations. Of course, more of them are needed at higher frequencies or larger aspect ratios. In this paper, the expansion limits are set to

$$M = N = 5 + (1 + 2\sqrt{2}q)ka_x. \tag{19}$$

This expression was found to produce a difference of less than 10^{-8} in the numerical results when M and N were incremented by 1. Hence, the authors have been able to reproduce Table I of Ref. 4 which has 7 decimal places, exactly. The radiation resistance and reactance are plotted for constant width in Figs. 2 and 3, respectively, and for constant area in Figs. 4 and 5, respectively. If the width is held constant and the aspect ratio increased to infinity, we obtain an infinite strip,[21] the radiation resistance and reactance of which are also shown in Figs. 2 and 3, respectively, using Eqs. (99) and (100) of Ref. 21. In Figs. 4 and 5, where the area is constant, we use radiation resistance and reactance of a circular piston as a reference, as given by Eqs. (13.117) and (13.118) of Ref. 11 in the form of the well-known Bessel and Struve functions.

At low frequencies

$$R_s \approx \rho c \frac{2q}{\pi} k^2 a_x^2, \quad ka_x < 0.5, \quad ka_y < 0.5, \tag{20}$$

$$X_s \approx \rho c \frac{2}{\pi}\left(\operatorname{arcsinh}(q) + q \operatorname{arccsch}(q) + \frac{1 + q^3 - \left(1+q^2\right)^{3/2}}{3q}\right) ka_x, \quad ka_x < 0.5, \quad ka_y < 0.5 \tag{21}$$

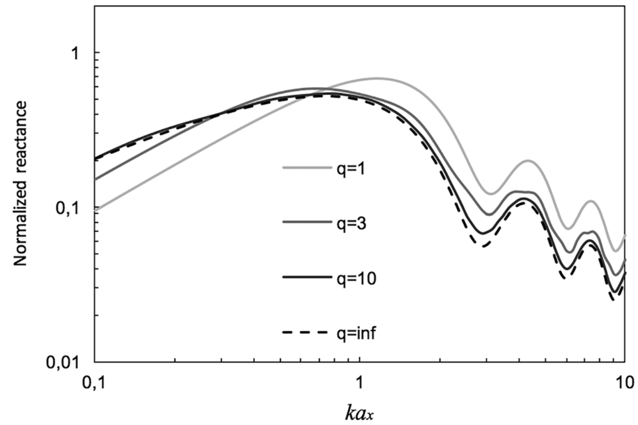

FIG. 3. Normalized specific radiation reactance $X_s/(\rho c)$ for various aspect ratios $q = a_y/a_x$ with constant half-width a_x.

where[20]

$$\operatorname{arcsinh}(q) = \ln\left(q + \sqrt{1+q^2}\right),$$
$$\operatorname{arccsch}(q) = \ln\left(q^{-1} + \sqrt{1+q^{-2}}\right). \tag{22}$$

All of these formulas are reciprocal with respect to q. In other words, they are still valid if we replace q with $1/q$, provided that we also replace a_x with a_y or qa_x. The low-frequency approximations of Stepanishen[13] are replicated if we substitute

$$\operatorname{arcsinh}(q) = \operatorname{arccosh}\left(\sqrt{1+q^2}\right),$$
$$\operatorname{arccsch}(q) = \operatorname{arccosh}\left(\sqrt{1+q^2}/q\right). \tag{23}$$

IV. ELLIPTIC PISTON

Because the radiation impedance of a square piston is very close to that of a circular one of the same area, it is interesting to compare the radiation impedance of a rectangular

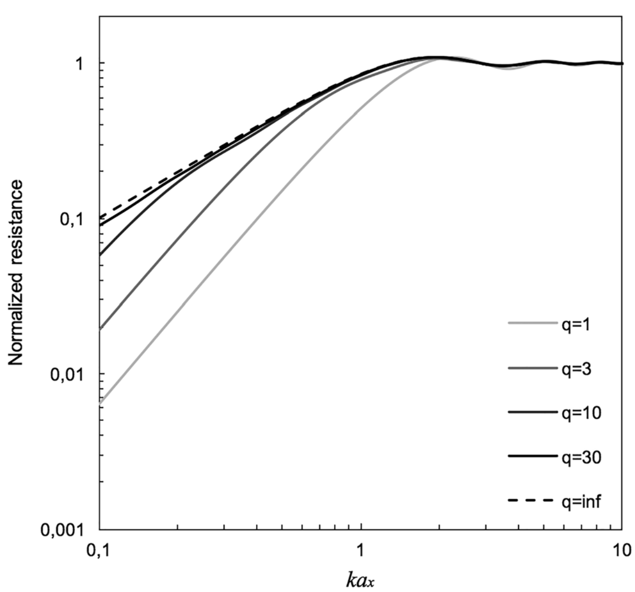

FIG. 2. Normalized specific radiation resistance $R_s/(\rho c)$ for various aspect ratios $q = a_y/a_x$ with constant half-width a_x.

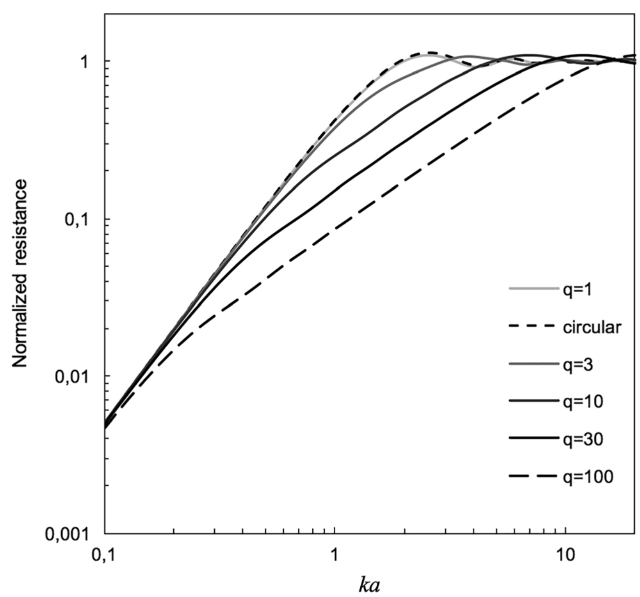

FIG. 4. Normalized specific radiation resistance $R_s/(\rho c)$ for various aspect ratios q with constant area $S = 4qa_x^2$, where $a = \sqrt{S/\pi}$.

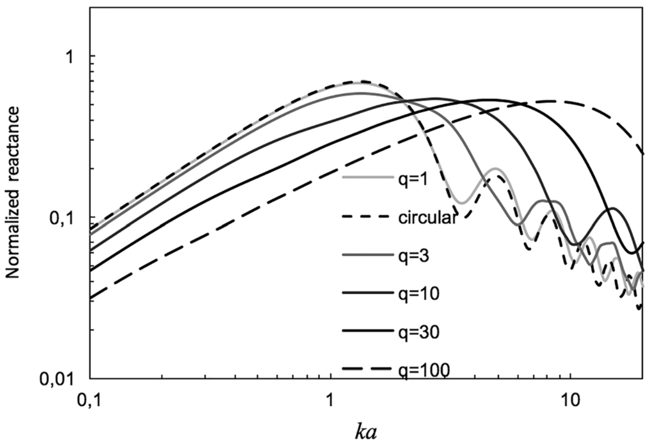

FIG. 5. Normalized specific radiation reactance $X_s/(\rho c)$ for various aspect ratios q with constant area $S = 4qa_x^2$, where $a = \sqrt{S/\pi}$.

piston with that of an elliptic one of the same area and aspect ratio. The directivity pattern of an elliptic piston[12] is given by

$$D(\theta,\phi) = 2\frac{J_1\left(k\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\sin\theta\right)}{k\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\sin\theta}. \tag{24}$$

Bouwkamp's impedance theorem[10,11] leads to the following integral expressions:

$$R_S = \frac{k^2\rho c S}{4\pi^2}\int_0^{2\pi}\int_0^{\pi/2}|D(\theta,\phi)|^2\sin\theta d\theta d\phi = \frac{2qk^2a_x^2}{\pi}\int_0^{\pi/2}\frac{1}{B^2(\phi)}\left(1 - \frac{J_1(2B(\phi))}{B(\phi)}\right)d\phi, \tag{25}$$

$$X_S = -j\frac{k^2\rho c S}{4\pi^2}\int_0^{2\pi}\int_{(\pi/2)+j0}^{(\pi/2)+j\infty}|D(\theta,\phi)|^2\sin\theta d\theta d\phi = \frac{2qk^2a_x^2}{\pi}\int_0^{\pi/2}\frac{\mathbf{H}_1(2B(\phi))}{B^3(\phi)}d\phi, \tag{26}$$

where

$$B(\phi) = ka_x\sqrt{\cos^2\phi + q^2\sin^2\phi} \quad \text{and} \quad S = \pi a_x a_y. \tag{27}$$

The authors found the version of Eq. (25) given in Ref. 12 to be in error as is also its solution. Hence correct solutions are given here for convenience. Expanding the Bessel J_1 and Struve $\mathbf{H}_1$ functions, while substituting $x = \sin\phi$, enables the integrals to be evaluated as follows:[22]

$$R_S = \frac{2q}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(ka_x)^{2m+2}}{(m+1)!(m+2)!}\int_0^1\frac{\left(1+(q^2-1)x^2\right)^m}{\sqrt{1-x^2}}dx = \frac{q}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m(ka_x)^{2m+2}}{(m+1)!(m+2)!}\times\sum_{n=0}^{m}\binom{m}{n}\frac{\Gamma(n+1/2)}{n!}\left(q^2-1\right)^n, \tag{28}$$

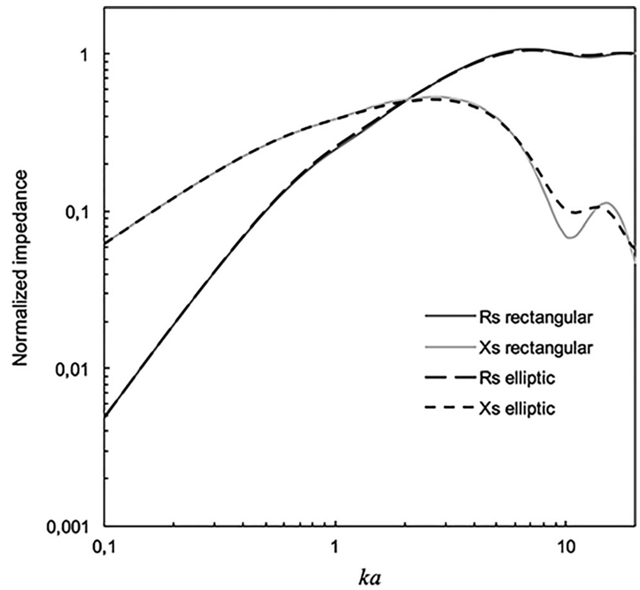

FIG. 6. Comparison of elliptic and rectangular pistons—normalized specific radiation resistance $R_s/(\rho c)$ and reactance $X_s/(\rho c)$ for aspect ratio $q = 10$ with constant area $S_{\text{ellip}} = \pi q a_x^2$ and $S_{\text{rect}} = 4qa_x^2$, where $a = \sqrt{S/\pi}$.

$$X_S = \frac{2q}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(ka_x)^{2m+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\frac{5}{2}\right)}\times\int_0^1\frac{\left(1+(q^2-1)x^2\right)^{m-1/2}}{\sqrt{1-x^2}}dx = q\sum_{m=0}^{\infty}\frac{(-1)^m(ka_x)^{2m+1}}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\frac{5}{2}\right)}\times {}_2F_1\left(\frac{1}{2},\frac{1}{2}-m;1;1-q^2\right), \tag{29}$$

where the resistance integral has been evaluated with help from the binomial theorem.[20]

At low frequencies

$$R_s \approx \rho c\frac{q}{2}k^2a_x^2, \quad ka_x < 0.5, \quad ka_y < 0.5, \tag{30}$$

$$X_s \approx \rho c\frac{16q}{3\pi^2}K\left(1-q^2\right)ka_x, \quad ka_x < 0.5, \quad ka_y < 0.5, \tag{31}$$

where K is the complete elliptic integral of the first kind. A comparison of the impedance of a rectangular and elliptic piston of the same area and aspect ratio $q = 10$ is shown in Fig. 6. From Eqs. (20) and (30) we can deduce that the resistances should be equal at very low frequencies. The reactances also appear to be close. It is at higher frequencies where we start to see some differences. Here the elliptical piston produces ripples which are smoother and less deep than those of the rectangular piston.

V. CONCLUSION

A set of expansions have been derived for evaluating the radiation resistance and reactance of a rectangular piston in an infinite baffle using the Fourier Green's function in rectangular coordinates in the Kirchhoff-Helmholtz boundary integral. They provide alternate expressions to those previously derived, which almost all involve numerical integration, and are hence a step closer to the simple expressions for a circular piston. They are highly amenable to numerical computation at high frequencies or large aspect ratios and yield simple low-frequency asymptotic expressions. However, it may be possible to simplify them even further through the use of recursion formulas and so this paper is intended to stimulate further research in the area.

The previous results of Bank and Wright,[2] which were obtained using a more computationally demanding integration, are reproduced in Figs. 4 and 5, albeit using 50 plot points per decade rather than just 10. The authors have also matched the tables of Ref. 4 exactly.

APPENDIX

A. Radiation resistance R_s

Substituting $s = \sin\varphi$ in Eq. (13) yields

$$R_s = \frac{4ka_x ka_y \rho c}{\pi^2} \int_0^1 \int_0^1 \mathrm{sinc}^2\left(ka_x t\sqrt{1-s^2}\right) \times \mathrm{sinc}^2(ka_y ts) \frac{tdtds}{\sqrt{1-t^2}\sqrt{1-s^2}}. \quad \text{(A1)}$$

The sinc squared terms in may be expanded using[20]

$$\mathrm{sinc}^2 x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1}}{(2n)!} x^{2n-2}, \quad \text{(A2)}$$

which gives

$$R_s = \frac{\rho c}{\pi^2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^{m+n} 2^{2m+2n} (ka_x)^{2m-1} (ka_y)^{2n-1}}{(2m)!(2n)!} \times \int_0^1 (1-s^2)^{m-3/2} s^{2n-2} ds \times \int_0^1 (1-t^2)^{-1/2} t^{2m+2n-3} dt. \quad \text{(A3)}$$

Applying the integral solutions[20]

$$\int_0^1 (1-s^2)^{m-3/2} s^{2n-2} ds = \frac{\Gamma\left(m-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)}{2\Gamma(m+n-1)}, \quad \text{(A4)}$$

$$\int_0^1 (1-t^2)^{-1/2} t^{2m+2n-3} dt = \frac{\sqrt{\pi}\,\Gamma(m+n-1)}{2\Gamma\left(m+n-\frac{1}{2}\right)}, \quad \text{(A5)}$$

while truncating the summation limits yields Eq. (15).

B. Radiation reactance X_s

The product of two sine squared terms in Eq. (14) can be reduced to a sum of cosine terms by first applying the identity $\sin^2 x = (1-\cos 2x)/2$ to give

$$X_s = \frac{4ka_x ka_y \rho c}{\pi^2} \int_0^{\pi/2} \int_1^{\infty} \frac{1-\cos(2ka_x t\cos\varphi)}{2(ka_x t\cos\varphi)^2} \times \frac{1-2\cos(2ka_y t\sin\varphi)}{2(ka_y t\sin\varphi)^2} \times \frac{t}{\sqrt{t^2-1}} dt d\varphi, \quad \text{(A6)}$$

and then applying $\cos a \cos b = (\cos(a+b) + \cos(a-b))/2$ while multiplying out the numerator terms to yield

$$X_s = \frac{\rho c}{2\pi^2 ka_x ka_y} \int_0^{\pi/2} \int_1^{\infty} \frac{1}{\cos^2\varphi \sin^2\varphi} \times \big(2 - 2\cos(2ka_x t\cos\varphi) - 2\cos(2ka_y t\sin\varphi) + \cos(2ka_x t\cos\varphi + 2ka_y t\sin\varphi) + \cos(2ka_x t\cos\varphi - 2ka_y t\sin\varphi)\big) \frac{1}{t^3\sqrt{t^2-1}} dt d\varphi, \quad \text{(A7)}$$

which is evaluated using the following identities:[22]

$$\int_1^{\infty} \frac{1}{t^3\sqrt{t^2-1}} dt = \frac{\pi}{4}, \quad \text{(A8)}$$

$$\int_1^{\infty} \frac{\cos at}{t^3\sqrt{t^2-1}} dt = \frac{\pi}{4}\bigg(1 - a^2 + a^3 J_0(a) - a^2 J_1(a) + a(a^2-1)\frac{\pi}{2}(J_1(a)\mathbf{H}_0(a) - J_0(a)\mathbf{H}_1(a))\bigg). \quad \text{(A9)}$$

Using the identity[23]

$$\int_0^a J_0(x)dx = a\left(J_0(a) + \frac{\pi}{2}(J_1(a)\mathbf{H}_0(a) - J_0(a)\mathbf{H}_1(a))\right), \quad \text{(A10)}$$

the cosine integral simplifies to

$$\int_1^{\infty} \frac{\cos at}{t^3\sqrt{t^2-1}} dt = \frac{\pi}{4}\left(aJ_0(a) - a^2 J_1(a) + (a^2-1) \times \left(\int_0^a J_0(x)dx - 1\right)\right) = \frac{\pi}{4}\left(1 - a^2 + \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \times \left(\frac{a^2}{(m+1)(2m+1)} + \frac{4m}{2m+1}\right) \times \left(\frac{a}{2}\right)^{2m+1}\right), \quad \text{(A11)}$$

where we have used the expansion[20]

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+\nu)!} \left(\frac{z}{2}\right)^{2m+\nu}. \quad \text{(A12)}$$

Using the integral of Eq. (A11), while substituting $s = \sin\varphi$ in Eq. (A7), and gathering all the terms into a single expansion yields

$$X_s = \frac{\rho c}{\pi k a_x k a_y}\int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\Bigg(\left(k a_x\sqrt{1-s^2}+k a_y s\right)^{2m+3} + |k a_x\sqrt{1-s^2}-k a_y s|^{2m+3}$$
$$-2\left(k a_x\sqrt{1-s^2}\right)^{2m+3} - 2(k a_y s)^{2m+3}\Bigg)\frac{1}{s^2(1-s^2)^{3/2}}ds, \qquad \text{(A13)}$$

which is split over two intervals of integration in order to remove the magnitude sign

$$X_s = \frac{\rho c}{\pi k a_x k a_y}(X_1 + X_2), \qquad \text{(A14)}$$

where

$$X_1 = \int_0^{a_x/\sqrt{a_x^2+a_y^2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\Bigg(\left(k a_x\sqrt{1-s^2}+k a_y s\right)^{2m+3} + \left(k a_x\sqrt{1-s^2}-k a_y s\right)^{2m+3}$$
$$-2\left(k a_x\sqrt{1-s^2}\right)^{2m+3} - 2(k a_y s)^{2m+3}\Bigg)\frac{1}{s^2(1-s^2)^{3/2}}ds, \qquad \text{(A15)}$$

$$X_2 = \int_{a_x/\sqrt{a_x^2+a_y^2}}^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\Bigg(\left(k a_x\sqrt{1-s^2}+k a_y s\right)^{2m+3} + \left(k a_y s - k a_x\sqrt{1-s^2}\right)^{2m+3}$$
$$-2\left(k a_x\sqrt{1-s^2}\right)^{2m+3} - 2(k a_y s)^{2m+3}\Bigg)\frac{1}{s^2(1-s^2)^{3/2}}ds. \qquad \text{(A16)}$$

Rearranging gives

$$X_1 = \int_0^{a_x/\sqrt{a_x^2+a_y^2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\Bigg(\left(k a_x\sqrt{1-s^2}\right)^{2m+3}$$
$$\times\left\{\left(1+\frac{a_y s}{a_x\sqrt{1-s^2}}\right)^{2m+3} + \left(1-\frac{a_y s}{a_x\sqrt{1-s^2}}\right)^{2m+3} - 2\right\} - 2(k a_y s)^{2m+3}\Bigg)\frac{1}{s^2(1-s^2)^{3/2}}ds, \qquad \text{(A17)}$$

$$X_2 = \int_{a_x/\sqrt{a_x^2+a_y^2}}^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\Bigg(\left(k a_x\sqrt{1-s^2}\right)^{2m+3}$$
$$\times\left\{\left(1+\frac{a_y s}{a_x\sqrt{1-s^2}}\right)^{2m+3} - \left(1-\frac{a_y s}{a_x\sqrt{1-s^2}}\right)^{2m+3} - 2\right\} - 2(k a_y s)^{2m+3}\Bigg)\frac{1}{s^2(1-s^2)^{3/2}}ds, \qquad \text{(A18)}$$

which can then be expanded using the binomial theorem[20]

$$\left(1\pm\frac{a_y s}{a_x\sqrt{1-s^2}}\right)^{2m+3} = \sum_{n=0}^{2m+3}(\pm 1)^n\binom{2m+3}{n}\left(\frac{a_y s}{a_x\sqrt{1-s^2}}\right)^n, \qquad \text{(A19)}$$

to yield

$$X_1 = \int_0^{a_x/\sqrt{a_x^2+a_y^2}} \sum_{m=0}^{\infty} \frac{2(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\Bigg(\left(k a_x\sqrt{1-s^2}\right)^{2m+3}\sum_{n=1}^{m+1}\binom{2m+3}{2n}\left(\frac{a_y s}{a_x\sqrt{1-s^2}}\right)^{2n}$$
$$-(k a_y s)^{2m+3}\Bigg)\frac{1}{s^2(1-s^2)^{3/2}}ds, \qquad \text{(A20)}$$

$$X_2 = \int_{a_x/\sqrt{a_x^2+a_y^2}}^1 \sum_{m=0}^{\infty} \frac{2(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\left(k a_x\sqrt{1-s^2}\right)^{2m+3}\left(\sum_{n=0}^{m}\binom{2m+3}{2n+1}\left(\frac{a_y s}{a_x\sqrt{1-s^2}}\right)^{2n+1} - 1\right)$$
$$\times\frac{1}{s^2(1-s^2)^{3/2}}ds, \qquad \text{(A21)}$$

where the pairs of expansions have been combined into even and odd expansions. Splitting the integrands into four separate integrals gives

$$X_1 = 2\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\left\{(ka_x)^{2m+3}\sum_{n=1}^{m+1}\binom{2m+3}{2n}\frac{a_y^{2n}}{a_x^{2n}}\int_0^{a_x/\sqrt{a_x^2+a_y^2}}(1-s^2)^{m-n}s^{2n-2}ds\right.$$
$$\left.-(ka_y)^{2m+3}\int_0^{a_x/\sqrt{a_x^2+a_y^2}}(1-s^2)^{-3/2}s^{2m+1}ds\right\}, \quad \text{(A22)}$$
$$X_2 = \sum_{m=0}^{\infty}\frac{2(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\left\{(ka_x)^{2m+3}\sum_{n=0}^{m}\binom{2m+3}{2n+1}\frac{a_y^{2n+1}}{a_x^{2n+1}}\int_{a_x/\sqrt{a_x^2+a_y^2}}^{1}(1-s^2)^{m-n-1/2}s^{2n-1}ds\right.$$
$$\left.-(ka_x)^{2m+3}\int_{a_x/\sqrt{a_x^2+a_y^2}}^{1}(1-s^2)^m s^{-2}ds\right\}, \quad \text{(A23)}$$

which has to be split up into six integrals as follows before evaluating because the $(m+1)$th term of the first expansion in n and the zeroth term of the second have different solutions from the rest

$$X_s = \frac{2\rho c}{\pi k a_x k a_y}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+3)(2m+1)m!(m+1)!}\left\{(ka_x)^{2m+3}(2m+3)\frac{a_y^{2m+2}}{a_x^{2m+2}}I_1(m)\right.$$
$$+(ka_x)^{2m+3}\sum_{n=1}^{m}\binom{2m+3}{2n}\frac{a_y^{2n}}{a_x^{2n}}I_2(m,n)-(ka_y)^{2m+3}I_3(m)+(ka_x)^{2m+3}(2m+3)\frac{a_y}{a_x}I_4(m)$$
$$\left.+(ka_x)^{2m+3}\sum_{n=1}^{m}\binom{2m+3}{2n+1}\frac{a_y^{2n+1}}{a_x^{2n+1}}I_5(m,n)-(ka_x)^{2m+3}I_6(m)\right\}, \quad \text{(A24)}$$

where the following integrals are evaluated with help from Eq. (A19) and the substitution of $t=(1-s^2)^{k-1/2}$ for I_3, I_4, and I_5, where $k=-1$, m, $m-n$, respectively,

$$I_1(m) = \int_0^{a_x/\sqrt{a_x^2+a_y^2}}(1-s^2)^{-1}s^{2m}ds = \ln\left(\frac{a_x}{a_y}+\sqrt{1+\frac{a_x^2}{a_y^2}}\right)-\sum_{p=1}^{m}\frac{1}{2p-1}\left(\frac{a_x}{\sqrt{a_x^2+a_y^2}}\right)^{2p-1}$$
$$=\frac{1}{(2m+1)\left(1+\frac{a_y^2}{a_x^2}\right)^{m+1/2}}\,{}_2F_1\left(1,m+\frac{1}{2};m+\frac{3}{2};\left(1+\frac{a_y^2}{a_x^2}\right)^{-1}\right), \quad \text{(A25)}$$

$$I_2(m,n) = \int_0^{a_x/\sqrt{a_x^2+a_y^2}}(1-s^2)^{m-n}s^{2n-2}ds = \sum_{p=0}^{m-n}\frac{(-1)^p}{2p+2n-1}\binom{m-n}{p}\left(\frac{a_x}{\sqrt{a_x^2+a_y^2}}\right)^{2p+2n-1}, \quad \text{(A26)}$$

$$I_3(m) = \int_0^{a_x/\sqrt{a_x^2+a_y^2}}(1-s^2)^{-3/2}s^{2m+1}ds = \sum_{p=0}^{m}\frac{(-1)^p}{2p-1}\binom{m}{p}\left(1-\left(\frac{a_y}{\sqrt{a_x^2+a_y^2}}\right)^{2p-1}\right), \quad \text{(A27)}$$

$$I_4(m) = \int_{a_x/\sqrt{a_x^2+a_y^2}}^{1}(1-s^2)^{m-1/2}s^{-1}ds = \int_0^{a_y/\sqrt{a_x^2+a_y^2}}(1-t^2)^{-1}t^{2m}dt = \ln\left(\frac{a_y}{a_x}+\sqrt{1+\frac{a_y^2}{a_x^2}}\right)-\sum_{p=1}^{m}\frac{1}{2p-1}\left(\frac{a_y}{\sqrt{a_x^2+a_y^2}}\right)^{2p-1}$$
$$=\frac{1}{(2m+1)\left(1+\frac{a_x^2}{a_y^2}\right)^{m+1/2}}\,{}_2F_1\left(1,m+\frac{1}{2};m+\frac{3}{2};\left(1+\frac{a_x^2}{a_y^2}\right)^{-1}\right), \quad \text{(A28)}$$

$$I_5(m,n) = \int_{a_x/\sqrt{a_x^2+a_y^2}}^{1}(1-s^2)^{m-n-1/2}s^{2n-1}ds = \sum_{p=0}^{n-1}\frac{(-1)^p}{2(p+m-n)+1}\binom{n-1}{p}\left(\frac{a_y}{\sqrt{a_x^2+a_y^2}}\right)^{2(p+m-n)+1}, \quad \text{(A29)}$$

$$I_6(m) = \int_{a_x/\sqrt{a_x^2+a_y^2}}^{1} (1-s^2)^m s^{-2} ds = \sum_{p=0}^{m} \frac{(-1)^p}{2p-1} \binom{m}{p} \left(1 - \left(\frac{a_x}{\sqrt{a_x^2+a_y^2}}\right)^{2p-1}\right), \tag{A30}$$

to yield

$$X_s = \frac{2\rho c}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (ka_x)^{2m+1}}{(2m+1)m!(m+1)!} \left\{ \frac{{}_2F_1\left(1, m+\frac{1}{2}; m+\frac{3}{2}, \frac{1}{1+q^2}\right)}{(2m+1)(1+q^{-2})^{m+1/2}} + \frac{{}_2F_1\left(1, m+\frac{1}{2}; m+\frac{3}{2}, \frac{1}{1+q^{-2}}\right)}{(2m+1)(1+q^{-2})^{m+1/2}} \right.$$
$$- \sum_{p=0}^{m} \frac{(-1)^p (q^{2m+2}+q^{-1})}{(2m+3)(2p-1)} \binom{m}{p} + \frac{1}{2m+3} \sum_{n=0}^{m} \left(\binom{2m+3}{2n} \sum_{p=0}^{m-n} \frac{(-1)^p q^{2n-1}}{(2p+2n-1)(1+q^2)^{p+n-1/2}} \binom{m-n}{p} \right.$$
$$\left.\left. + \binom{2m+3}{2n+3} \sum_{p=0}^{n} \frac{(-1)^p q^{2p+2m+1}}{(2p+2m-2n-1)(1+q^2)^{p+m-n-1/2}} \binom{n}{p} \right) \right\}, \tag{A31}$$

where $q = a_y/a_x$ is the aspect ratio and we have used the second (Hypergeometric) solutions of the integrals in Eqs. (A25) and (A28). The reason for this is that at high frequencies, where the first (logarithmic) solution needs to have many terms in the expansion, errors arise when subtracting the two parts of the solution which have almost the same value. However, the first solutions will be used to obtain the low-frequency asymptotic approximation of Eq. (21). Alternatively, using[20]

$$\sum_{k=0}^{m} \frac{(-1)^k}{2k-1} \binom{m}{k} = -(2m+3)(2m+1) \int_0^1 (1-s^2)^{1/2} s^{2m+1} ds = -\frac{\sqrt{\pi} m!}{\Gamma\left(m+\frac{1}{2}\right)}, \tag{A32}$$

together with Eq. (A2), and re-ordering the expansions, while truncating their limits, leads to the simplified expression of Eq. (16).

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